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Totally geodesic surfaces in hyperbolic 3-manifolds

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Totally geodesic surfaces in hyperbolic 3-manifolds

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This thesis investigates the topology and geometry of hyperbolic 3-manifolds containing totally geodesic surfaces. In chapter 2 we find algebraic invariants associated to several families of hyperbolic 3-manifolds with totally geodesic boundary. In chapter 3 we describe examples of hyperbolic knot complements in rational homology 3-spheres containing closed embedded totally geodesic surfaces.

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Chapter 1

Introduction and background

This thesis investigates the geometry and topology of hyperbolic three-manifolds containing totally geodesic surfaces. A theme in the study of closed three-manifolds since its inception has been that much useful information can be gained by cutting a three-manifold apart along embedded surfaces and relating the topology of the resulting manifolds-with-boundary to that of the original. W.P. Thurston extended this philosophy to the geometric realm in his proof of hyperbolization for Haken three-manifolds. An irreducible 3-manifold M — one in which every embedded 2-sphere bounds a ball — is *Haken* if it contains an embedded essential surface S of positive genus, that is, one whose inclusion $S \hookrightarrow M$ induces a monomorphism at the level of fundamental group. Thurston proved that every *atoroidal* Haken manifold (one without an essential torus) admits a hyperbolic structure. The proof proceeds by first cutting open such a manifold along an essential surface and then using the description of the deformation space of hyperbolic structures on the resulting manifolds-with-boundary, and the action on this deformation space of the gluing map which identifies the boundaries, in order to find compatible hyperbolic structures on each side.

A particularly simple hyperbolic structure on a manifold with boundary is one in which the boundary is *totally geodesic*, that is with vanishing principal curvatures. Any manifold-with-boundary M obtained by cutting open an atoroidal Haken manifold admits such a hyperbolic structure, if in addition it is *acylindrical* — that is, if every copy of $S^1 \times I$ properly embedded in M is homotopic into ∂M — however, there is only one such structure in the entire positive–dimensional deformation space of hyperbolic structures on M! This is due to the fact that a structure with totally geodesic boundary naturally extends to a hyperbolic structure on the *double* of M across its boundary, and Mostow rigidity now applies to show that such structures are unique. This uniqueness property makes it unsurprising that it seems somewhat rare for a surface in a hyperbolic Haken manifold to be totally geodesic (this is equivalent to the structures on either side having hyperbolic structures with matching totally geodesic boundaries). One may suspect that hyperbolic Haken manifolds possessing totally geodesic surfaces have some special topological features; one reflection of this is the "Menasco-Reid" conjecture.

Conjecture (Menasco-Reid, [35])**.** *No hyperbolic knot complement in S^3 contains a closed embedded totally geodesic surface.*

Work of several authors has established that many families of hyperbolic knot complements in S^3 possess topological features which obstruct the existence of closed embedded totally geodesic surfaces, however this is certainly not true in general, and the conjecture remains very much open. In fact it is only recently that hyperbolic knot complements with no totally geodesic surfaces at all, embedded or immersed, closed or cusped, have been exhibited [11]. A chapter of this thesis is dedicated to giving examples that show the analogous conjecture is false if one only requires that the knot complements occur in manifolds with the rational homology of S^3.

The thesis is organized as follows. The first chapter is largely devoted to exposition, introducing notation and our particular perspective on the subject. This combines material from many well–known sources, notably [50], [38], and [45]. The first section of this chapter introduces hyperbolic manifolds with totally geodesic boundary. We begin with the naive definition that a hyperbolic manifold with geodesic boundary is simply a manifold with an atlas of chart maps to hyperbolic half spaces with isometric overlaps, and move quickly to relate this to Kleinian groups via the standard tools of the developing map $\mathcal{D}$ and holonomy representation $\mathcal{H}$. The key theorem states that for a hyperbolic manifold M with totally geodesic boundary, $\mathcal{D}$ induces an isometric embedding of M onto the convex core of $\mathbb{H}^n/\mathcal{H}(\pi_1 M)$. Next we introduce hyperbolic *orbifolds* with totally geodesic boundary and describe the analogous theory. In particular, the above theorem applies verbatim in this case, replacing "manifold" with "orbifold", and "π_1" with "π_1^{orb}" (which may have torsion, in contrast to the manifold case).

In the last section of Chapter 1 we prove a version of the Poincaré polyhedron theorem for gluings of *truncated* polyhedra which produce hyperbolic manifolds with totally geodesic boundary. The Poincaré polyhedron theorem gives sufficient conditions under which the identification space of a polyhedron in $\mathbb{H}^n$ by a scheme which pairs sides by isometries is a hyperbolic orbifold, and describes the fundamental group of this orbifold in terms of the subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ generated by the side–pairing isometries. We define a truncated polyhedron to be a hyperbolic polyhedron with a specified collection of *external* sides which do not abut each other, and abut the other, *internal* sides at right angles. Concerning *internal side–pairings* of such polyhedra — that is, schemes for pairing only the internal sides — we prove the following analog of the Poincaré polyhedron theorem (a more precise statement is in Theorem 3 below).

Theorem. *If an internal side–pairing of a truncated polyhedron P yields a complete hyperbolic orbifold P^* with geodesic boundary, then P^* is the convex core of the quotient of $\mathbb{H}^n$ by the group generated by the side–pairing isometries. This group is subject only to the natural "edge cycle" relations.*

This is a fundamental tool which we use in the remainder of the thesis to describe examples of hyperbolic 3-orbifolds with totally geodesic boundary. While we do not know of our particular formulation appearing previously in the literature, it would certainly not surprise any expert in Kleinian groups, and an ad hoc argument in [42] displays all essential features of the proof. The exposition in this section is designed to closely parallel Maskit's definitive treatment in [32].

Chapter 2 is devoted to an extended study of several infinite families of hyperbolic orbifolds with totally geodesic boundary appearing in work of Paoluzzi-Zimmermann [42] and Frigerio [17]. In [42], Paoluzzi-Zimmerman construct an infinite family O_n, $n \geq 3$, of compact orbifolds with geodesic boundary. Notably, Thurston's "tripus" manifold (Ch 3 of [50]) is a threefold branched cover of O_3. In §2.1 we identify explicit generators in $\mathrm{PSL}_2(\mathbb{C})$ for the Kleinian groups associated to O_3 (Theorem 4) and O_4 (Theorem 5), and obtain as a corollary a description of generators for the Kleinian group associated to the Tripus (Corollary 1). In the following section we consider a family O'_g, $g \geq 2$, of noncompact finite–volume hyperbolic orbifolds with compact geodesic boundary covered by manifolds constructed by R. Frigerio in [17]. For each $g \geq 2$, we obtain an explicit description of the Kleinian group associated to O'_g in Theorem 6, as well as its presentation. Frigerio's orbifold O'_2 has a threefold branched cover which is the complement in a genus two handlebody of the "Adams-Reid knot" of [6]. There knot complements in S^3 were constructed by gluing this knot to the tripus along their boundaries, and it was shown that the resulting surface was not totally geodesic. Our perspective supplies another proof by inspection of the totally geodesic boundary structures on either side, since they are nonisometric.

Section 2.3 contains a discussion of the doubles of these manifolds and a study of certain invariants of their associated Kleinian groups. Among these invariants are the *trace field* and *invariant trace field* (a thorough discussion of these may be found in [30]), which have recently been shown in some cases to obstruct fibering, and in others to obstruct the existence of totally geodesic surfaces [11]. Here we explicitly determine these invariants for Paoluzzi-Zimmermann's orbifolds O_3 and O_4 (Theorem 7) and each of Frigerio's orbifolds O'_{g+1} (Theorem 8). In the final section, we determine the cusped manifolds O_∞ and O'_∞ which are geometric limits of the families $\{O_n\}$ and $\{O'_{g+1}\}$, respectively, and explicitly describe their associated Kleinian groups (Theorems 9 and 10, respectively). The geometric limit supplies a convenient uniform platform from which to examine certain features of the entire family, which perspective is useful in the following chapter.

Chapter 3 is devoted to the production of counterexamples to the "Menasco-Reid conjecture for rational homology spheres" mentioned above. The author has previously published this material in [16]. Various strengthenings of the Menasco-Reid conjecture are known to be false, for instance there exist links in S^3 with hyperbolic complement containing a closed embedded totally geodesic surface. The first such was exhibited by Menasco-Reid in [35], and Leininger constructed k–component links with this property for any $k \geq 2$, as well as hyperbolic knot complements in S^3 containing surfaces with arbitrarily small principal curvatures, [26]. Our first theorem of chapter 3 answers a "warm–up" question.

Theorem (Theorem 12 below)**.** *There exist infinitely many hyperbolic rational homology spheres containing closed embedded totally geodesic surfaces.*

A hyperbolic rational homology sphere is a closed hyperbolic manifold with the same rational homology as S^3 — in particular, it has first Betti number equal to 0. The primary interest of this theorem is that it shows that a totally geodesic surface does not necessarily contribute any homology to the ambient manifold. The second main theorem gives examples of hyperbolic *knot complements* in rational homology spheres containing closed embedded totally geodesic surfaces.

Theorem (Theorem 11 below)**.** *There exist infinitely many hyperbolic knot complements in rational homology spheres containing closed embedded totally geodesic* *surfaces.*

A hyperbolic knot complement in a rational homology sphere is a finite volume one-cusped hyperbolic manifold with the property that the result of *Dehn filling* on some slope in a torus cross–section of the cusp yields a rational homology sphere. Dehn filling of a cusped hyperbolic manifold is the process whereby the cusp (which we recall is homeomorphic to $T^2 \times (0, \infty)$, where T^2 is the two-dimensional torus) is truncated at some torus cross section and the resulting manifold is glued to a solid torus (homeomorphic to $D^2 \times S^1$) via a homeomorphism between their boundaries, yielding a closed manifold into which the original hyperbolic manifold embeds as the complement of the core of the filling torus. The topological type of the filled manifold is determined by the isotopy class of simple closed curve which is identified with the boundary of a disk $D^2 \times \{p\}$ — isotopy classes of simple closed curves on the torus are called *slopes*.

The theorem above provides infinitely many counterexamples to the direct generalization of the Menasco-Reid conjecture to hyperbolic knot complements in rational homology spheres and, to the extent that knot complements in rational homology spheres resemble their counterparts in S^3, may be regarded as evidence against the conjecture. We remark that homology seems to be a fairly coarse measure of topological complexity of a three-manifold, since in fact there exist hyperbolic 3-manifolds with the same *integral* homology as S^3.

In the final chapter we discuss various further questions which have arisen in the course of our investigations and make some conjectures.

1.1 Hyperbolic manifolds with geodesic boundary

Define the hyperbolic n–space to be the unique complete, simply connected Riemannian manifold of constant sectional curvature -1. We will do computations in the *upper half–space model*,

$$\mathbb{H}^n = \{\ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\ \}$$

with Riemannian metric $\langle\ ,\ \rangle_H = \left(\frac{1}{x_n}\right)\langle\ ,\ \rangle_E$, where $\langle\ ,\ \rangle_E$ is the standard Euclidean bilinear form

$$\langle\, (v_1, \ldots, v_n), (w_1, \ldots, w_n)\,\rangle_E = v_1 w_1 + \ldots + v_n w_n$$

on the tangent space to any point. The *angle* between tangent vectors $\mathbf{v}$ and $\mathbf{w}$ is defined as usual to be the value in $[0, \pi)$ of θ so that $\cos\theta = \frac{\langle\,\mathbf{v},\mathbf{w}\,\rangle}{\|\mathbf{v}\|\,\|\mathbf{w}\|}$, where $\|\mathbf{v}\| = \sqrt{\langle\,\mathbf{v},\mathbf{v}\,\rangle}$. Hyperbolic space is naturally compactified by a *sphere at infinity*, in this model consisting of $(\mathbb{R}^{n-1} \times \{0\}) \cup \{\infty\}$ topologized so that it is homeomorphic to S^{n-1}. This is denoted S^{n-1}_∞. The *geodesic hyperplanes* of $\mathbb{H}^n$—that is, the isometric embeddings of $\mathbb{H}^{n-1}$—consist in the upper half–space model of Euclidean spheres and planes intersecting $\mathbb{R}^{n-1} \times \{0\}$ at right angles.

The isometry group $\mathrm{Isom}(\mathbb{H}^n)$ is generated by reflections in the geodesic hyperplanes. Its action evidently extends continuously to S^{n-1}_∞, since reflection in a geodesic hyperplane preserves $(\mathbb{R}^{n-1} \times \{0\}) \cup \{\infty\}$. A nontrivial orientation–preserving isometry of $\mathbb{H}^{n-1}$ is classified as *elliptic*, *parabolic*, or *hyperbolic* according to whether it fixes a point of $\mathbb{H}^n$, no points of $\mathbb{H}^n$ and a unique point of S^{n-1}_∞, or no points of $\mathbb{H}^n$ and two of S^{n-1}_∞, respectively. (These are the only possibilities.) In three dimensions, the orientation–preserving subgroup is naturally isomorphic to $\mathrm{PSL}_2(\mathbb{C})$, whose action on $\mathbb{H}^3$ extends its action on S^2_∞ by Möbius transformations. The full isometry group of $\mathbb{H}^3$ is isomorphic to $\mathrm{PSL}_2(\mathbb{C}) \rtimes \langle\, r\,\rangle$, where r is the reflection in the vertical plane $\{0\} \times \mathbb{R}^2$. Although the trace of an element of $\mathrm{PSL}_2(\mathbb{C})$ is not well-defined, the square of its trace is. A nontrivial element of $\mathrm{PSL}_2(\mathbb{C})$ is elliptic, parabolic, or loxodromic if the square of its trace is in the interval $[0, 4)$, equal to 4, or otherwise, respectively.

An *open hyperbolic half space* U is one component in $\mathbb{H}^n$ of the complement of a geodesic hyperplane H. Its *frontier* (that is, its topological boundary as a subset of $\mathbb{H}^n$) is H, and we define a *hyperbolic half space* B to be the closure in $\mathbb{H}^n$ of an open half space U:

$$B = \overline{U} = U \cup H.$$

We define a *hyperbolic n–manifold with totally geodesic boundary* to be a manifold-with-boundary M which admits an atlas of charts $\{\,\phi_\alpha\colon\ U_\alpha \to B_\alpha\,\}$ to hyperbolic half spaces bounded by geodesic hyperplanes $H_\alpha \subset \mathbb{H}^n$, so that the collection $\{U_\alpha\}$ covers M, each chart map satisfies $\phi_\alpha(U_\alpha \cap \partial M) = \phi_\alpha(U_\alpha) \cap H_\alpha$, and overlap maps are restrictions of isometries. M is *orientable* if overlap maps may be chosen to preserve orientation. Such an atlas gives a Riemannian metric on M of constant curvature -1.

For a hyperbolic n-manifold with totally geodesic boundary M and a fixed choice of $x \in M$ and chart $\phi\colon\ U \to \mathbb{H}^n$ around x, a *developing map* which is a local isometry from the universal cover $\widetilde{M}$ to $\mathbb{H}^n$ may be constructed by analytically continuing ϕ along paths emanating from x. $\mathcal{D}\colon\ \widetilde{M} \to \mathbb{H}^n$ induces a *holonomy representation* $\mathcal{H}\colon\ \pi_1 M \to \Gamma < \mathrm{Isom}(\mathbb{H}^n)$, satisfying the following equivariance relation: for $x \in \widetilde{M}$ and $\gamma \in \pi_1(M)$,

$$\mathcal{D}(\gamma.x) = \mathcal{H}(\gamma).\mathcal{D}(x).$$

We say M is *complete* if $\mathcal{D}\colon\ M \to \mathbb{H}^n$ is a covering map onto its image. This coincides with completeness of the Riemannian metric on M.

Lemma 1. *For M a complete finite-volume hyperbolic manifold with totally geodesic boundary, the developing map and holonomy representation have the following properties:*

- *$\mathcal{D}$ maps $\widetilde{M}$ isometrically onto the intersection of a countable collection of hyperbolic half-spaces bounded by mutually disjoint geodesic hyperplanes.*
- *$\mathcal{H}$ maps $\pi_1 M$ faithfully onto a group of isometries acting discontinuously on $\mathbb{H}^n$.*

Proof. By completeness, $\mathcal{D}$ is a covering map onto a convex $\mathcal{H}$-equivariant subset of $\mathbb{H}^n$. Since $\mathcal{D}(\widetilde{M}) \subset \mathbb{H}^n$ is convex it is simply connected; hence $\mathcal{D}$ is an isometry.

Each component of $\partial\widetilde{M}$, covering a component of ∂M, maps under the developing map to a subset of a geodesic hyperplane H in $\mathbb{H}^n$. Since $\mathcal{D}$ is a local isometry, the image is an open subset $U \subseteq H$. I claim U is closed as well, and hence all of H. A point x in the closure is approached by a convergent sequence of points in U, which pulls back and down to a convergent sequence in ∂M. Since M and hence ∂M is complete, this sequence converges to a point in ∂M, and the appropriate preimage of this point maps to x under $\mathcal{D}$. This proves the claim.

Let $\{H_\alpha\}$ denote the boundary components of $\widetilde{M}$, and by a slight abuse of notation the geodesic hyperplanes of $\mathbb{H}^3$ which are their images under the developing map. For each component of ∂M, there is an H_α for each conjugate of the image of its fundamental group under the map induced by the inclusion $\partial M \hookrightarrow M$. For each α, the image of $\widetilde{M}$ is contained in one of the hyperbolic half–planes bounded by H_α; denote this by B_α. Thus $\mathcal{D}$ maps $\widetilde{M}$ to an open subset of $\cap B_\alpha$; this subset is also closed by an argument like the above, so that $\mathcal{D}(M) = \cap B_\alpha$. That the H_α do not intersect follows from the fact that distinct boundary components of $\widetilde{M}$ are disjoint, since each such is mapped onto an H_α.

It follows from the equivariance property above the statement of the lemma that $\Gamma = \mathcal{H}(\pi_1 M)$ acts discontinuously on $\mathcal{D}(\widetilde{M}) = \cap B_\alpha$, a convex subset of $\mathbb{H}^n$. A Γ–equivariant retraction may be defined $\mathbb{H}^n \to \mathcal{D}(\widetilde{M})$, taking a point to its nearest point in $\mathcal{D}(\widetilde{M})$. This implies that the action of Γ on all of $\mathbb{H}^n$ is discontinuous, since any compact subset of $\mathbb{H}^n$ which intersects infinitely many of its Γ–translates would project to a compact subset of $\mathcal{D}(\widetilde{M})$ with the same property, contradicting discontinuity of the action on $\mathcal{D}(\widetilde{M})$. $\square$

A Kleinian group is simply a discrete group of isometries. It is easily seen that discreteness is equivalent to discontinuity of the action on $\mathbb{H}^n$; hence for a hyperbolic manifold M with totally geodesic boundary, $\Gamma = \mathcal{H}(\pi_1 M)$ is a Kleinian group. A fundamental object associated to any Kleinian group Γ is its *limit set* $L_\Gamma \subseteq S^{n-1}_\infty$, which is the closure of the set of loxodromic fixed points of Γ. If Γ is *nonelementary* (that is, if L_Γ consists of more than two points), the limit set is the minimal closed nonempty Γ–invariant subset of S^{n-1}_∞. Its convex hull in $\mathbb{H}^n$ is the minimal closed Γ–invariant subset of $\mathbb{H}^n$, whose quotient is the *convex core* of $\mathbb{H}^n/\Gamma$; that is, the minimal convex submanifold which carries the fundamental group. The complement of L_Γ in S^{n-1}_∞ is the *domain of discontinuity* Ω, the largest subset on which Γ acts discontinuously. In fact, Γ acts discontinuously on $\mathbb{H}^n \cup \Omega$.

Theorem 1. *Let M be a complete finite volume hyperbolic manifold with totally geodesic boundary, and let Γ be the image of $\pi_1 M$ under the holonomy representation. The developing map induces an isometry from M to the convex core of $\mathbb{H}^n/\Gamma$.*

Proof. Let $D_\alpha \subset S^{n-1}_\infty$ be the open disk in the complement of $\overline{B_\alpha}$, with notation as above, where the closure is taken in the closed unit ball $\mathbb{H}^n \cup S^{n-1}_\infty$. Then $\overline{\cap B_\alpha} \cap S^{n-1}_\infty = S^{n-1}_\infty - (\cup D_\alpha)$. Since M contains the convex core of $\mathbb{H}^n/\Gamma$, the limit set L_Γ is contained in $S^{n-1}_\infty - (\cup D_\alpha)$. Now suppose there is some $x \in \Omega \cap \overline{\cap B_\alpha}$. Then by discontinuity there is an $\epsilon_0 > 0$ for which a closed (Euclidean) ball of radius less than ϵ_0 about x intersects its Γ–translates only finitely many times. By taking ϵ small enough one can insure that the closed ϵ neighborhood is disjoint from its translates. Thus this ball injects into $\mathbb{H}^n/\Gamma$, and its intersection with $\cap B_\alpha$ injects into M. But the intersection with $\cap B_\alpha$ has infinite volume in $\mathbb{H}^n$, a contradiction to the fact that M has finite volume. Thus the limit set is all of $\overline{\cap B_\alpha} \cap S^{n-1}_\infty$, and the theorem follows.

$\square$

Kojima has proved the following theorem in all dimensions. In dimension three it is an immediate corollary of the above and Ahlfors' finiteness theorem.

Theorem (Kojima)**.** *The boundary of a complete finite volume hyperbolic n–manifold with totally geodesic boundary is a finite volume hyperbolic $n-1$–manifold.*

1.2 Hyperbolic orbifolds with geodesic boundary

Any hyperbolic isometry of finite order fixes a point of $\mathbb{H}^n$ by the Brouwer fixed-point theorem, since it preserves the convex hull of a point orbit, which is homeomorphic to a ball. Thus for a hyperbolic manifold with boundary M, the associated Kleinian group $\Gamma = \mathcal{H}(\pi_1 M)$ is *torsion–free*, since elements of the fundamental group act by covering transformations (in particular, without fixed points) on a convex subset of $\mathbb{H}^n$. It is frequently convenient to broaden the class of spaces under consideration to include such objects as the quotient of a hyperbolic manifold by a discrete group of isometries which does not necessarily act freely. For this reason we introduce hyperbolic *orbifolds* with totally geodesic boundary, which roughly speaking are spaces locally modeled on quotients of half spaces by finite groups of isometries.

The theory of hyperbolic orbifolds with boundary follows the contours of the theory for manifolds, but at each step the extra local structure adds complication.

Below we first define a smooth orbifold with boundary and then say what it means to have a hyperbolic structure.

Definition. *A smooth orbifold O with totally geodesic boundary is a second–countable, paracompact topological space X_O together with an atlas of charts $\phi_i \colon U_i \to \widetilde{U}_i/\Gamma_i$ satisfying the following conditions*

1. *For each i, $U_i \subset X_O$ is open and ϕ_i is a homeomorphism of U_i onto the quotient of an open subset $\widetilde{U}_i$ of a half space of $\mathbb{R}^n$ by a finite group Γ_i of diffeomorphisms preserving $\widetilde{U}_i$.*

2. *The collection $\{U_i\}$ covers X_O and is closed under intersection.*

3. *If $U_i \subset U_j$, there is an injective homomorphism $f_{ij} \colon \Gamma_i \to \Gamma_j$ and an f_{ij}–equivariant smooth embedding $\widetilde{\phi}_{ij} \colon \widetilde{U}_i \to \widetilde{U}_j$ (that is $\widetilde{\phi}_{ij}(\gamma.x) = f_{ij}(\gamma).\widetilde{\phi}_{ij}(x)$ for $x \in U_i$ and $\gamma \in \Gamma_i$) so that the following diagram commutes*

$$\begin{tikzcd}
\widetilde{U}_i \arrow[r, "\widetilde{\phi}_{ij}"] \arrow[d] & \widetilde{U}_j \arrow[d] \\
\widetilde{U}_i/\Gamma_i \arrow[r, "\phi_{ij}"] & \widetilde{U}_j/\Gamma_j \\
U_i \arrow[u, "\phi_i"] \arrow[r] & U_j \arrow[u, "\phi_j"']
\end{tikzcd}$$

   Here ϕ_{ij} is the map induced by $\widetilde{\phi}_{ij}$, taking the equivalence class $[x]$ of $x \in \widetilde{U}_i$ to the equivalence class of $\widetilde{\phi}_{ij}(x)$ under the action of $f_{ij}(\Gamma_i)$ and then further projecting to the equivalence class under the action of all of Γ_j. The bottom horizontal arrow is inclusion.

Two such atlases generate the same orbifold structure if their union satisifies the criteria above.

This was written to precisely extend Thurston's original definition ([50], §13.2) to orbifolds with boundary (cf. page 23 of [14]). We take the *boundary* ∂O of such an orbifold O to be the set of points which admit a chart map to the equivalence class of a point in the boundary of a half space, and the *interior* of O is $O - \partial O$. For a point $x \in O$ and a chart $\phi\colon U \to \widetilde{U}/\Gamma$ around x, the *local group* Γ_x at x is the subgroup $\mathrm{Stab}_\Gamma(\widetilde{x})$ for some preimage $\widetilde{x} \in \widetilde{U}$ of $\phi(x)$. By condition (3) and the observations above, this determines a well–defined conjugacy class of subgroups of $\mathrm{Isom}(\mathbb{H}^n)$. The *order* of x is the order of the local group at x. If this is one, x is said to be *regular*. Otherwise x is *singular*; the set of all such points is the *singular locus*.

An orbifold with boundary is *hyperbolic* with *totally geodesic boundary* if it has an atlas in which all charts map to quotients of open subsets of hyperbolic half spaces by discrete groups of isometries, and the embeddings ϕ_{ij} of condition 3 are isometries. The fact that isometries of open subsets of $\mathbb{H}^n$ extend uniquely to global isometries implies certain simplifications in this case. The following lemma relates a condition which more closely resembles the "overlap map" definition for hyperbolic manifolds, and sometimes replaces condition 3 in the definition of hyperbolic orbifold (eg. in [45], §13.2).

Lemma. *An orbifold is hyperbolic with totally geodesic boundary if and only if it has a covering of chart maps $\{U_i, \phi\}$ mapping to quotients of open subsets of hyperbolic half spaces by hyperbolic isometries, and if $U_i \cap U_j \neq \emptyset$ then for $x \in \widetilde{U}_i$ and $y \in \widetilde{U}_j$ with $\phi_j\phi_i^{-1}(\Gamma_i x) = \Gamma_j y$, there is an isometry g taking x to y which lifts the overlap map $\phi_j\phi_i^{-1}$ in a neighborhood of x.*

Proof. Suppose O is hyperbolic with totally geodesic boundary. Then if $U_i = U_j \cap U_k$, we may take g to be the isometry of $\mathbb{H}^n$ extending $\widetilde{\phi}_{ij} \circ \widetilde{\phi}_{ik}^{-1}$. This carries $\widetilde{U}_k$ to intersect $\widetilde{U}_j$ in $\widetilde{\phi}_{ij}(\widetilde{U}_i)$ and lifts $\phi_j \circ (\phi_k|_{U_i})^{-1}\colon\ \phi_k(U_i) \to \phi_j(U_i)$, hence satisfying the statement of the lemma.

If O satisfies the statement of the lemma and $U_i \subset U_j$, then fixing $x \in \widetilde{U}_i$ we claim that the isometry g supplied by the hypotheses of the lemma plays the role of both $\widetilde{\phi}_{ij}$ and f_{ij} of condition 3, letting $f_{ij}(\gamma) = g\gamma g^{-1}$ for $\gamma \in \Gamma_i$. This follows directly from the equivariance condition and the fact mentioned above the lemma. $\square$

Coverings of orbifolds may be defined as well; these are covers in the usual sense on the complement of the singular locus, but may branch at points of the singular locus.

Definition. *An orbifold covering $\widetilde{O} \to O$ consists of an onto map $X_{\widetilde{O}} \to X_O$ which respects the orbifold structure in the following sense: every point $x \in O$ has a chart neighborhood U for which every component V of $p^{-1}(U)$ is a chart neighborhood such that the following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "\psi"'] \arrow[d, "p"'] & \widetilde{U}/\Gamma' \arrow[d, "\pi"] \\ U \arrow[r, "\phi"] & \widetilde{U}/\Gamma. \end{tikzcd}$$

Here $\phi\colon U \to \widetilde{U}/\Gamma$ and $\psi\colon V \to \widetilde{U}/\Gamma'$ are chart maps, $\Gamma' \subset \Gamma$ is a subgroup, and π is the natural projection.

As is the case with manifolds, each orbifold has a *universal* orbifold cover; that is, a cover $\pi\colon \widetilde{O} \to O$ with the universal property that π factors through any other cover of O. Thus for any orbifold cover $p\colon O' \to O$, there is a covering

$\pi' \colon \widetilde{O} \to O'$ so that the following diagram commutes.

$$\begin{tikzcd}
\widetilde{O} \arrow[dd, "\pi"] \arrow[dr, "\pi'"] & \\
& O' \arrow[dl, "p"] \\
O &
\end{tikzcd}$$

We note that $\widetilde{O}$ has little to do with the universal cover of the underlying topological space X_O. Indeed, in the next chapter we consider examples for which the underlying topological space is S^3, but the orbifold universal cover is $\mathbb{H}^3$. The orbifold universal cover is regular, in the sense that two preimages of a point $x \in O$ are related by an isometry of $\widetilde{O}$ which commutes with π. The group of such isometries is called the *orbifold fundamental group* of O, denoted $\pi_1^{orb}(O)$, in analogy to the case of a manifold M, where the deck transformation group of the universal cover is isomorphic to $\pi_1(M)$.

A standard construction of the universal orbifold cover collects information about lifts of paths to $\mathbb{H}^n$, in much the same way that the universal cover of a manifold M may be defined as the set of homotopy classes rel endpoints of paths in M starting at a specified base point. Because of the local structure, in the orbifold case the lift of a path through $x \in O$ is not determined simply by a choice of chart ϕ around x if the local group at x is nontrivial. For this reason we define a *path* based at $x \in O$ to consist not only of a continuous map $I \to X_O$ based at x, but also a set of charts covering the image, a choice of lift for each chart, and a selection of gluing maps between adjacent charts which preserve lifts. $\widetilde{O}$ is then defined to be the set of paths based at a fixed basepoint x_0 up to an equivalence relation which includes a notion of homotopy rel endpoints. The details of this construction are in Chapter 13.3 of [45], in the empty boundary case, and it extends to the geodesic boundary case without requiring further comment.

With this construction, $\pi_1^{orb}(O)$ is defined to be the group of closed paths at x_0, with the group operation being concatenation, as usual. The covering projection $\widetilde{O} \to O$ and the action of $\pi_1^{orb}(O)$ on $\widetilde{O}$ by deck transformations follow easily. Furthermore, a developing map $\mathcal{D}\colon \widetilde{O} \to \mathbb{H}^n$ is easily defined by analytic continuation. A holonomy representation $\mathcal{H}\colon \pi_1^{orb} \to \mathrm{Isom}(\mathbb{H}^n)$ which commutes with $\mathcal{D}$ is also defined. Each boundary component of $\widetilde{O}$ maps to an open subset of a geodesic hyperplane, and hence that $\mathcal{D}$ maps onto an open subset of some intersection of geodesic half spaces of $\mathbb{H}^n$.

As in the manifold case, we say O is *complete* if the developing map is a covering onto its image. In this case, results analogous to Lemma 1 and Theorem 1 hold. We collect them in the following theorem.

Theorem 2. *Suppose O is a finite volume hyperbolic orbifold with boundary. Then the following holds.*

1. *$\mathcal{D}$ maps $\widetilde{O}$ isometrically onto the intersection of a countable collection of hyperbolic half spaces bounded by mutually disjoint geodesic hyperplanes.*

2. *$\mathcal{H}$ maps π_1^{orb} isomorphically onto a Kleinian group Γ, and $\mathcal{D}$ induces an isometry between O and the convex core of $\mathbb{H}^n/\Gamma$.*

Proof. The proof is analogous to the proofs of Lemma 1 and Theorem 1. □

1.3 The Poincaré polyhedron theorem

A fundamental tool for giving examples of hyperbolic orbifolds, especially in low dimensions, is the *Poincaré polyhedron theorem*, which describes the relationship between an orbifold obtained by gluing in pairs the faces of a convex polyhedron and the group generated by the isometries realizing those face–pairings for some embedding of the polyhedron in $\mathbb{H}^n$. This theorem was originally stated by

Poincaré for two–dimensional polygons, but his proof apparently had a gap. The gap was definitively filled by Maskit, who gave a proof of the analogous theorem in three dimensions as well [31]. Morokuma proved a version in all dimensions [39], and a fully general statement and proof were given in [32]. Here we parallel this treatment to prove a version of the theorem for manifolds with totally geodesic boundary. Once notation has been established, and with the aid of a few lemmas, this follows quickly from the without–boundary version.

Definition. *A hyperbolic polyhedron is the (nonempty) intersection of open hyperbolic half spaces bounded by a locally finite collection of geodesic hyperplanes.*

For every hyperbolic polyhedron P there is a unique minimal countable collection of geodesic hyperplanes H_i and associated half spaces B_i such that the closure $\bar{P} = \bigcap B_i$, with the additional property that for each i, $\bar{P} \cap H_i$ contains an open subset of H_i. The *sides* of P are the interiors in the H_i of subsets $\bar{P} \cap H_i$, for such a minimal collection of H_i. Each of these are themselves polyhedra in H_i, so polyhedra of one lower dimension, being of the form

$$\mathrm{Int}\left(\bigcap_{j \neq i} H_i \cap B_j\right).$$

The side of a side of P we will call a *edge* of P. This is a codimension two convex polyhedron of the form $H_i \cap H_j$ for two distinct hyperplanes in the above collection. In general we will call *faces* of P the convex polyhedra obtained by iterating the operation of taking sides as above.

It follows from the definition above that a edge R of a polyhedron P is the intersection of exactly two sides S_i and S_j. The geodesic hyperplanes H_i and H_j containing S_i and S_j divide $\mathbb{H}^n$ into four components, one of which contains P. For a point $x \in S_i \cap S_j$, define $\mathbf{v_i}$ and $\mathbf{v_j}$ to be the unique normals in $T_x\mathbb{H}^n$ to the tangent planes to H_i and H_j, respectively, chosen so that they point away from

P. Define the *dihedral angle* of R to be $\theta = \pi - \alpha$, where α is the angle between $\mathbf{v_i}$ and $\mathbf{v_j}$. This does not depend on the choice of x since H_i and H_j are totally geodesic.

In the context of hyperbolic manifolds with totally geodesic boundary we will work with *truncated* hyperbolic polyhedra.

Definition. *A truncated hyperbolic polyhedron is a hyperbolic polyhedron whose sides have been divided into two classes, internal and external, such that each external side abuts only internal sides, and the dihedral angle of each edge bounding an external side is* $\pi/2$.

We call any lower–dimensional face of P *external* if it is contained in an external side and *internal* otherwise, and similarly for points of P. An internal side S of P is itself a truncated polyhedron, with external sides corresponding to external edges of P bounding S. To each truncated polyhedron P we associate a polyhedron EP, the *expansion* of P, obtained by intersecting only the hyperbolic half spaces corresponding to geodesic hyperplanes containing the internal sides of P. It follows from the definitions that P is a convex subpolyhedron of EP, and each face of EP contains a unique internal face of P. The relationship between P and EP is the main tool we use to transfer facts about boundaryless hyperbolic manifolds to the geodesic boundary case. The following lemma records the key aspects of this relationship.

Lemma 2. *The components of* $EP - P$ *are in one-to-one correspondence with external sides of* P*, and the component associated to an external side* S *is homeomorphic to* $S \times (0, \infty)$*, with second coordinate given by distance to* S*.*

Proof. Consider an external side S of P and the collection $\{S_1, S_2, \ldots\}$ of sides which it intersects (all of which are internal, by definition). For each i let H_i be the geodesic hyperplane containing S_i, and let B_i be the hyperbolic half space bounded by H_i which contains P. The geodesic hyperplane H containing S intersects the H_i perpendicularly and divides $(\bigcap B_i)$ into two components. Define C_S to be the component of $(\bigcap B_i) - H$ which does not contain P. We claim that C_S is a component of $EP - P$.

To see this, consider the polyhedron P' which is the intersection of all half spaces from the construction of P, except for B. If C_S is not entirely contained in P', then there is a geodesic hyperplane H' containing a side S' of P not among the S_i specified above, which also contains a side of $C_S \cap P'$. Since S' is a side of P, in particular it is contained in $(\bigcap B_i) \cap B$. Thus H' has points of intersection with $\bigcap B_i$ on either side of H; however since S' does not intersect S, $H' \cap H$ is excluded from $(\bigcap B_i) \cap H = S$. But since H' contains the geodesics between its points of intersection with $\bigcap B_i$ on either side of H, this is contradicts convexity of H. Thus C_S is entirely contained in $P' \subset EP$. It is easily seen that P separates C_S from the remainder of EP, since $S \subset P$ separates C_S from the remainder of $\bigcap B_i$, which contains the remainder of $EP - P$. This proves the claim.

To establish the one-to-one correspondence between components of $EP - P$ and external sides of P, it only remains to note that each $x \in EP - P$ is separated from P by some external side S, by definition of EP. Thus $x \in C_S$.

In order to prove that C_S is homeomorphic to $S \times (0, \infty)$, we introduce the retraction of $r_S \colon \mathbb{H}^n \to H$ which maps a point to the nearest point of H to it. For any $x \in H$, $r_S^{-1}(x)$ is the geodesic through x perpendicular to H. Since each H_i intersects H perpendicularly, B_i is precisely the preimage under this retraction of its intersection with H. It follows that C_S is one component of $r_S^{-1}(S) - H$. For any fixed $d > 0$, consider the collection $C_S(d)$ consisting of points of C_S which are distance d away from S. This maps onto S under r, since for any point x of S one can go out a distance of d in the appropriate direction along the geodesic through x to find a preimage in $C_S(d)$. And the mapping is 1–1, since there is only one way to do this.

$\square$

Our analog of the Poincaré polyhedron theorem concerns manifolds with totally geodesic boundary obtained by identifying the internal faces of a truncated polyhedron in pairs by an *internal side–pairing*.

Definition (c.f. [32], IV.H.2)**.** *An internal side–pairing for a truncated hyperbolic polyhedron P is a collection of isometries $\{ g_S \colon S \in \mathcal{S} \}$, indexed by the collection $\mathcal{S}$ of all internal sides of P, with the property that for each $S \in \mathcal{S}$,*

1. *there is a side S' in $\mathcal{S}$ with $g_S(S) = S'$;*
2. *the isometries g_S and $g_{S'}$ satisfy the relation $g_{S'} = g_S^{-1}$;*
3. *$P \cap g_S(P) = \emptyset$; and*
4. *for each internal side S of P, $g_S \colon S \to S'$ is an isometry of truncated polyhedra—that is, g_S takes internal (resp. external) edges of P bounding S to internal (external) edges bounding S'.*

This mirrors Maskit's definition of a side–pairing of a polyhedron, except that in this definition the external sides are not identified. Also new is condition (4) above, from which it follows that the sets of internal and external faces of P are preserved by the side–pairing. An internal side–pairing of a truncated polyhedron P generates an equivalence relation on the elements of $\bar{P}$, where $x \in \bar{P}$ and $y \in \bar{P}$ are equivalent if there is some sequence of side–pairing isometries $g_{S_1}, \ldots, g_{S_n}$ such that $(g_{S_1} \circ \cdots \circ g_{S_n})(x) = y$. Call the identification space P^*, equipped with the quotient topology so that the projection $p \colon \bar{P} \to P^*$ is continuous. We call $x^* = p(x)$ *external* if x is external and *internal* otherwise. By condition (4) above this does not depend on the choice of $x \in p^{-1}(x^*)$, and we denote by ∂P^* the collection of external points of P^*.

Lemma 3. *An internal side–pairing for a truncated hyperbolic polyhedron P induces a side–pairing (in the sense of Maskit) of its expansion EP.*

Proof. Let S and S' be internal sides of P, paired by g_S. By definition of EP there are sides ES and ES' of EP containing S and S'. Since the side–pairing is internal, for each external side R of S (of the form $R = E \cap S$, where E is an external side of P) there is an external side R' of S' with $g_S(R') = R$. Applying Lemma 2 to S and S', it follows that g_S takes the component $C_{R'}$ of $ES' - S'$ isometrically to $C_R \subset ES - S$. This is because each point of $C_{R'}$ is determined by a point $r' \in R'$ and a distance $d \in (0, \infty)$, and since g_S is an isometry the image of this point is the point determined by the point $g_S(r') = r \in R$ and d. Hence g_S takes ES' isometrically to ES.

□

Since an internal side–pairing takes internal edges to internal edges, the equivalence relation above induces an equivalence relation on the set of internal edges. The equivalence classes we call *internal edge cycles*. If an internal edge cycle of P is finite, then it may be enumerated as $\{e_1, \ldots, e_n\}$, by choosing an initial edge e_1 and internal side S_1 containing e_1. S_1 is paired to an internal side S_1' by the side–pairing g_{S_1}; let $e_2 = g_{S_1}(e_1)$, and denote by S_2 the other side of P containing e_2. Now iterate this process until arriving back at the pair (e_1, S_1). This gives a sequence $\{e_1, \ldots, e_n\}$ of edges, a sequence $\{S_1, \ldots, S_n\}$ of sides, and a sequence $\{g_1, \ldots, g_n\}$ of internal side–pairings whose salient feature is that $h = g_n \circ \cdots \circ g_1$ preserves the pair (e_1, S_1). The *dihedral angle sum* of such a finite internal edge cycle is

$$\theta([e_i]) = \theta_1 + \ldots + \theta_n,$$

where θ_i is the dihedral angle of the edge e_i.

Definition. *An internal side–pairing of a truncated polyhedron P is admissible if it satisfies the following additional conditions.*

1. *For each x in $\bar{P}$, $p^{-1}(p(x))$ is finite.*

2. *For each edge cycle $[e_i]$, $\theta([e_i]) = 2\pi/t$ for some $t \in \mathbb{N}$, and the associated transformation h satisfies the edge cycle relation $h^t = 1$.*

Admissible internal side pairings yield hyperbolic orbifolds with totally geodesic boundary. We prove this below using the following construction due to Maskit (cf. [32], IV.H.7). For an admissible internal side–pairing of a truncated polyhedron P define a group G^* by the presentation

$$G^* = \langle\, \{g_S \,|\, \text{S an internal side of P}\} \mid \{g_S = g_{S'}^{-1}\} \cup \{h^t = 1\,\}\, \rangle.$$

By admissibility of the side–pairing, the subgroup G of $\mathrm{Isom}(\mathbb{H}^n)$ generated by the side–pairing isometries satisfies the relations of G^*, and we call the induced epimorphism $\sigma\colon\ G^* \to G$. Give G^* the discrete topology, and consider the quotient of $G^* \times P$ by the equivalence relation generated by making (g_1^*, x_1) and (g_2^*, x_2) equivalent if there is a side–pairing f with $f(x_1) = x_2$ and $g_2^* = g_1^* \circ f^{-1}$ as words in G^*. Define X^* to be this quotient, equipped with the quotient topology.

There is a natural map $q\colon\ X^* \to P^*$ induced by projection to the second factor followed by p, and another $r\colon\ X^* \to \mathbb{H}^n$ given by $r(g^*, x) = \sigma(g^*)(x)$. Furthermore, G^* acts on X^* by homeomorphisms via $g^*.(h^*, x) = (g^*h^*, x)$ and r commutes with the action of G^* under σ. This displays P^* as a hyperbolic orbifold with totally geodesic boundary, with $q\colon\ X^* \to P^*$ its universal orbifold cover and $r\colon\ X^* \to \mathbb{H}^n$ the developing map. This is the content of the lemma below. Its statement parallels Maskit's Lemma IV.H.12.

Lemma 4. *For an admissible internal side–pairing of a truncated polyhedron P, every point $x^* \in P^*$ has a neighborhood U whose inverse image $q^{-1}(U)$ is a disjoint union of relatively compact open sets U_α. Furthermore, for each α, $r|U_\alpha$ is a homeomorphism onto a convex subset of a hyperbolic half space with bounding* *hyperplane H, and $r(U_\alpha \cap q^{-1}(\partial P^*)) = r(U_\alpha) \cap H$.*

Proof. The key point is that an admissible internal side pairing of P induces a side pairing of its expansion EP which satisfies Maskit's conditions (i) through (vi) on pp. 73–75 of [32]. Thus Maskit's contruction applies to $G^* \times EP$, constructing a space EX^* into which X^* embeds, and to which Lemma IV.H.12 may be applied directly. The maps $q\colon X^* \to P^*$ and $r\colon X^* \to \mathbb{H}^n$ factor through the embedding $X \hookrightarrow EX$, and X is invariant under the G–action on EX.

For each $x^* \in P^*$, Lemma IV.H.12 furnishes a neighborhood, which we will call $\widetilde{U}$, of x in EP^* with the property that preimages $\widetilde{U}_\alpha$ in EX^* are relatively compact and map homeomorphically to convex subsets of $\mathbb{H}^n$. $\widetilde{U}$ is constructed from δ–neighborhoods of preimages $p^{-1}(x) \in E\bar{P}$, where δ is less than half the minimum of the distances of each such preimage from the others and the sides of EP in which it is not contained. If $x^* \in P - \partial P$, taking δ to be smaller than the distance of each preimage to the collection of external sides, in addition to the above requirements, yields a subneighborhood $U \subset \widetilde{U} \cap P^*$ with the same properties.

If $x^* \in \partial P^*$, then taking δ to be smaller than the minimum of the distances of each preimage to the other external sides, as well as the above requirement yields a neighborhood $\widehat{U} \subset \widetilde{U}$ of x^* in EP^*, and we claim that $U = \widehat{U} \cap P^*$ satisfies the statement of the lemma. To see this, enumerate the preimages $x_1, \ldots, x_k \in \bar{P}$ of x^*, each of which is in the closure of a unique internal side S_i contained in a geodesic hyperplane H_i. For $i > 1$, let $g_i^* \in G*$ identify x_i to x_1, and let $g_i = \sigma g_i^*$. The set $p^{-1}(\widehat{U}) \subset E\bar{P}$ is a disjoint union of components $\widehat{V}_i$ around the x_i, and the proof Lemma IV.H.12 asserts that each preimage $\widehat{U}_\alpha$ under q is a G^*–translate of the neighborhood

$$\widehat{U}_1 = (1, \widehat{V}_1) \cup \left(\bigcup_{i=2}^{k} ((g_i^*)^{-1}, \widehat{V}_i) \right),$$

which is a neighborhood of $(1, x_1)$ in EX^*. Furthermore, $r|\widehat{U}_\alpha$ is a homeomorphism, for $\widehat{U}_1$ in particular onto a neighborhood of x_1. In each case g_i must take H_i to H since it is a composition of internal side–pairings, each of which preserves the collection of external sides. Thus

$$U_1 = \widehat{U}_1 \cap X^* = (1, \widehat{V}_1 \cap \bar{P}) \cup \left(\bigcup_{i=2}^{k} ((g_i^*)^{-1}, \widehat{V}_i \cap \bar{P}) \right)$$

maps to $r(\widehat{U}_1) \cap B$, where B is the half space determined by H, with boundary points taken to H. Since r is G–equivariant, it follows that all such neighborhoods have the same property.

$\square$

The Poincaré polyhedron theorem is simply the consequence of Lemma 4 when the quotient space P^* is complete.

Theorem 3. *If an admissible internal side–pairing of a truncated polyhedron P yields a complete quotient orbifold P^* with geodesic boundary, then σ is an isomorphism of G^* with a Kleinian group G and the map r above is an isometry of X^* with the convex hull of $\sigma(G^*)$ in $\mathbb{H}^n$. In particular, r induces an isometry of P^* with the convex core of $\mathbb{H}^n/\sigma(G^*)$.*

Chapter 2

Examples

2.1 The tripus and friends

In this section we apply the Poincaré polyhedron theorem to compute some Kleinian groups associated to a family $\{ O_n \}$, $n \geq 3$, of compact hyperbolic orbifolds with totally geodesic boundary constructed by Paoluzzi-Zimmermann [42]. For each n, O_n is the orbifold with geodesic boundary indicated in Figure 2.1. For k relatively prime to n, Paoluzzi–Zimmerman describe a hyperbolic manifold $M_{n,k}$ with geodesic boundary which is an n–fold branched cyclic cover of O_n. In particular $M_{3,1}$ is the "tripus" constructed by Thurston in his notes [50] (see Ch. 3). Here we compute Kleinian groups associated to O_3 and O_4, and as a corollary of the description of O_3 obtain a description of generators for the tripus group.

In Figure 2.1(a) is a picture of the underlying topological space, the three-dimensional ball B^3, with the tangle pictured corresponding to the singular locus. The label n by each component of the singular locus denotes a cone angle of $2\pi/n$ around it. For each $n \geq 3$, Paoluzzi-Zimmermann construct O_n by an internal side–pairing of the "truncated tetrahedron" in Figure 2.1(b), so called because it is obtained from a tetrahedron by cutting off open neighborhoods of the vertices.

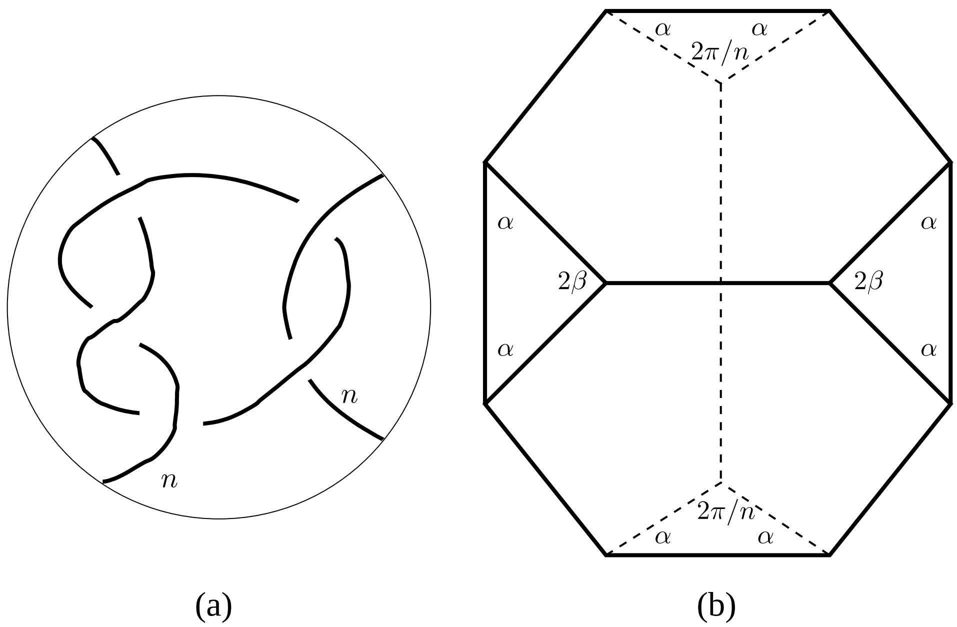


Figure 2.1: The orbifold O_n

In the truncated tetrahedron the external faces consist precisely of the triangular ones, hence all of their edges must have dihedral angle $\pi/2$ and the dihedral angles of the remaining edges are prescribed by the vertex angles of the triangular faces as shown.

In [42] values for α and β depending on n are prescribed so that the following internal side–pairing is realized by isometries. Denote by h the rotation about the "back" dashed edge which takes the right back face to the left one; then any point on this edge is an edge cycle. Call x the identification given by first rotating the top front face clockwise by $2\pi/3$ and then taking it to the bottom front face by a rotation about the horizontal edge. In order for x to be realizable as an isometry, all internal edges of the front hexagonal faces must have identical length. Paoluzzi-Zimmermann show this occurs if and only if

$$\cot\beta = \cot\alpha + \frac{\cos(2\pi/n)}{\cos\alpha\sin\alpha}. \tag{2.1}$$

The edge cycle associated to a point on any internal edge but the dashed one includes all others, and the dihedral angle sum is $4\alpha + 2\beta$. We have

Theorem (Paoluzzi-Zimmermann). *For each $n \geq 3$ there are unique α_n and β_n satisfying $4\alpha_n + 2\beta_n = 2\pi/n$ and (1) above, and an isometric embedding of P_n in $\mathbb{H}^3$ with these dihedral angles. The group Γ_n generated by x and h satisfies*

$$\Gamma_n \cong \langle\ x, h \mid h^n = (hxhx^{-2})^n = 1\ \rangle.$$

The convex core O_n of $\mathbb{H}^3/\Gamma_n$ is a hyperbolic orbifold with totally geodesic boundary homeomorphic to the tangle of Figure 2.1(a), with singular locus taken to the tangle strings.

The proof that Γ_n has the above presentation follows from the Poincaré polyhedron theorem. Here we choose a particular embedding of P_n and write down the generators of Γ_n for $n = 3$ and 4. To make this choice, we note that P_n has commuting reflective symmetries; in Figure 2.1(b) these are reflections in planes perpendicular to the page through the dashed vertical edge and the bold horizontal edge. We choose the embedding so that these planes are the vertical geodesic hyperplanes over $i\mathbb{R}$ and $\mathbb{R}$, respectively. These planes intersect in the geodesic between 0 and ∞, which intersects P_n in a geodesic arc connecting the dashed vertical edge of Figure 2.1 to the horizontal edge, perpendicular to both. The embedding is determined by choosing the bottom endpoint of this arc to occur at $(0,1) \in \mathbb{C} \times \mathbb{R}^+$.

Next we assemble geometric information about P_n to be used to compute representations for h and x. Paoluzzi-Zimmerman compute that the sides paired by x have internal and external edges of lengths B and C, respectively, satisfying

$$\cosh B = \frac{\cosh C}{\cosh C - 1} \qquad\qquad \cosh C = \frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}$$

(See [42], p. 116.) Additionally, we compute the length L of a perpendicular bisector from the center of one of these sides to the midpoint of an internal edge, as pictured in Figure 2.2.

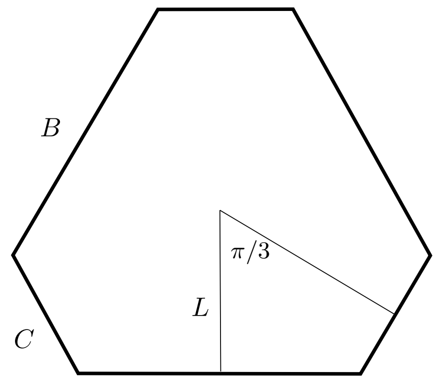


Figure 2.2: An internal face of P_n

In the hyperbolic metric the pictured hexagon has all right angles, since external and internal edges of P_n meet at right angles. Thus the quadrilateral in the figure with side labeled L has all right angles but at the labeled vertex. Note that the side of this quadrilateral opposite L has length $C/2$. A standard hyperbolic trigonometric formula for such quadrilaterals (see eg. [45], Thrm 3.5.7) yields

$$\cosh L = \frac{\cosh(C/2)}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}\sqrt{\frac{1}{2}(\cosh C + 1)} = \frac{2\cos(\pi/n)}{\sqrt{3}\sin\alpha}.$$

We also computate the distance between "front" and "back" internal edges of the truncated tetrahedron. With our embedding of P_n, this is the length M of the arc on the geodesic between 0 and ∞ in the cross–section of P_n through the hyperplane H lying above $i\mathbb{R}$, as pictured in Figure 2.3.

Let A be the length of the pictured edge of $H \cap P_n$. (*Warning*: although our notation is otherwise consistent with that of [42], there A refers to a different quantity.) This edge lies in an external side of P_n as the geodesic arc between a vertex with angle $2\pi/n$ and the midpoint of the opposite edge of the side. Using the

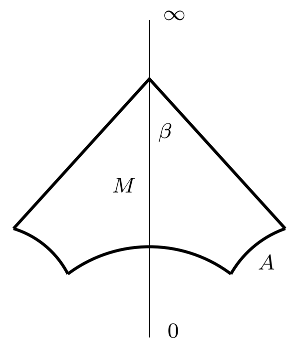


Figure 2.3: A cross–section of P_n

hyperbolic law of cosines (see eg. [45], Thrm 3.5.4) to find A and the trigonometric formula for quadrilaterals to compute M yields

$$\cosh A = \frac{\cos\alpha}{\sin(\pi/n)} \qquad\qquad \cosh M = \frac{\cosh A}{\sin\beta}$$

In order to write down matrices for h and x, we require the following standard facts. Elliptic elements of $\mathrm{PSL}_2(\mathbb{C})$ fixing $\pm i$ are of the form $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$, where a and b are real numbers with $a^2 + b^2 = 1$. Furthermore, such an element rotates by θ in a right–handed direction around its axis oriented from i to $-i$, where $e^{i\theta/2} = a + bi$. We call this element ρ_θ. The corresponding elliptic which rotates by θ around an axis from 1 to -1 is $\sigma_\theta = \left(\begin{smallmatrix} a & -bi \\ -bi & a \end{smallmatrix}\right)$. Finally, the hyperbolic element which translates upward along the axis between 0 and ∞ with translation length $\lambda > 0$ is of the form $\tau_\lambda = \left(\begin{smallmatrix} r & 0 \\ 0 & 1/r \end{smallmatrix}\right)$, where r is the root of the polynomial $x^4 - 2(\cosh\lambda)x^2 + 1$ which is greater than 1. Solving this polynomial gives

$$r = \sqrt{\cosh\lambda + \sinh\lambda} = e^{\lambda/2}.$$

If $\lambda < 0$, we take $\tau_\lambda := \tau_{|\lambda|}^{-1}$ to be the hyperbolic element translating downward by $|\lambda|$ along the geodesic from 0 to ∞. (A *hyperbolic* element of $\mathrm{PSL}_2(\mathbb{C})$ is simply one with a real trace.)

For the side–pairing isometries of our embedding of P_n, we have $h = \rho_{2\pi/n}$, since the bottom edge of P_n (the dashed edge of Figure 2.1(a)) lies on the geodesic from i to $-i$ by our choice of embedding of P_n. We write $x = e^2 f$, where $e = \tau_M\,\sigma_\beta\,\tau_M^{-1}$ is the elliptic rotating around the top edge of P_n and f rotates the appropriate internal face of P_n clockwise by $2\pi/3$. In Figure 2.1(b), this is the top hexagonal face with all bold edges, and we have $f = e^{-1}\,\tau_{M-L}\,\rho_{2\pi/3}\,\tau_{M-L}^{-1}\,e$. This gives

$$
\begin{aligned}
x &= e\,\tau_{M-L}\,\rho_{2\pi/3}\,\tau_{M-L}^{-1}\,e && (2.2)\\
&= \tau_M\,\sigma_\beta\,\tau_M^{-1}\,\tau_{M-L}\,\rho_{2\pi/3}\,\tau_{M-L}^{-1}\,\tau_M\,\sigma_\beta\,\tau_M^{-1} && (2.3)\\
&= \tau_M\,\sigma_\beta\,\tau_{-L}\,\rho_{2\pi/3}\,\tau_{-L}^{-1}\,\sigma_\beta\,\tau_M^{-1} && (2.4)
\end{aligned}
$$

One may in principle use the above to compute matrices for h and x for any $n \geq 3$. We carry this out for $n = 3$ and $n = 4$ below.

Theorem 4. *O_3 is isometric to the convex core of $\mathbb{H}^3/\Gamma_3$, where $\Gamma_3 = \langle\, h, x\,\rangle < \mathrm{PSL}_2(\mathbb{C})$ with h and x given by*

$$
h = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \qquad x = \begin{pmatrix} \frac{\sqrt{3}}{4} + i\frac{(\sqrt{3}+1)\sqrt{-2+3\sqrt{3}}}{4\sqrt{2}} & x_{12} \\ x_{21} & \frac{\sqrt{3}}{4} + i\frac{(\sqrt{3}+1)\sqrt{-2+3\sqrt{3}}}{4\sqrt{2}} \end{pmatrix}
$$

where

$$
x_{12} = \frac{1}{4\sqrt{3}}\left[4 + 3\sqrt{3} + 2\sqrt{-2+3\sqrt{3}} - i\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)(2 + \sqrt{-2+3\sqrt{3}})\right]
$$
$$
x_{21} = \frac{-1}{4\sqrt{3}}\left[4 + 3\sqrt{3} - 2\sqrt{-2+3\sqrt{3}} + i\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)(2 - \sqrt{-2+3\sqrt{3}})\right]
$$

A presentation for this Kleinian group is $\Gamma_3 = \langle\, h, x \mid h^3 = (hxhx^{-2})^3 = 1\,\rangle$.

Proof. The presentation for Γ_3 is a direct consequence of Paoluzzi-Zimmermann's theorem in the case $n = 3$. In order to compute the representation, we collect the relevant geometric data from above in this case. We have $\alpha = \pi/12$ and $\beta = \pi/6$ (see [42], p. 117), and hence

$$\cosh C = 3 + 2\sqrt{3} \qquad \cosh B = \frac{3+\sqrt{3}}{4}$$

$$\cosh L = 2\sqrt{\frac{2+\sqrt{3}}{3}} = \frac{2+2\sqrt{3}}{\sqrt{6}}$$

$$\cosh A = \frac{1+\sqrt{3}}{\sqrt{6}} \qquad \cosh M = \frac{2+2\sqrt{3}}{\sqrt{6}}$$

The matrices we use from above are now

$$h = \rho_{2\pi/3} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \qquad \sigma_\beta = \sigma_{\pi/6} = \begin{pmatrix} \frac{\sqrt{3}+1}{2\sqrt{2}} & -i\frac{\sqrt{3}-1}{2\sqrt{2}} \\ -i\frac{\sqrt{3}-1}{2\sqrt{2}} & \frac{\sqrt{3}+1}{2\sqrt{2}} \end{pmatrix}$$

$$\tau_{-L} = \begin{pmatrix} \left(\frac{2+2\sqrt{3}}{\sqrt{6}} + \sqrt{\frac{5+4\sqrt{3}}{3}}\right)^{-1/2} & 0 \\ 0 & \left(\frac{2+2\sqrt{3}}{\sqrt{6}} + \sqrt{\frac{5+4\sqrt{3}}{3}}\right)^{1/2} \end{pmatrix} = \tau_M^{-1}$$

We note that $\sqrt{5+4\sqrt{3}} = (\sqrt{3}+1)\sqrt{\frac{-2+3\sqrt{3}}{2}}$. Using this fact, we rewrite the matrix above.

$$\tau_{-L} = \begin{pmatrix} \left(\left(\frac{\sqrt{3}+1}{\sqrt{6}}\right)(2+\sqrt{-2+3\sqrt{3}})\right)^{-1/2} & 0 \\ 0 & \left(\left(\frac{\sqrt{3}+1}{\sqrt{6}}\right)(2+\sqrt{-2+3\sqrt{3}})\right)^{1/2} \end{pmatrix} = \tau_M^{-1}$$

The matrix for x is now easily computed using the formula of equation (2.4). $\square$

The veracity of these formulas may be easily verified using one's favorite computer algebra program.

Corollary 1. *The tripus is isometric to the convex core of* $\mathbb{H}^3/\Gamma_{3,1}$*, where* $\Gamma_{3,1} = \langle x_1, x_2, x_3 \rangle < \Gamma_3$ *with* $x_i = h^i x h^{-i-1}$ *for each* i*. As elements of* $\mathrm{PSL}_2(\mathbb{C})$*, the* x_i

are given by

$$x_1 = \begin{pmatrix} \frac{\sqrt{3}+1}{2} & \left(\sqrt{2}+i(\sqrt{3}+1)\right)\frac{1-\sqrt{-2+3\sqrt{3}}}{2\sqrt{6}} \\ \left(\sqrt{2}-i(\sqrt{3}+1)\right)\frac{1-\sqrt{-2+3\sqrt{3}}}{2\sqrt{6}} & \frac{\sqrt{3}+1}{2} \end{pmatrix}$$

$$x_2 = \begin{pmatrix} -\frac{\sqrt{3}+1}{2}+\frac{\sqrt{-2+3\sqrt{3}}}{4}-i\frac{\sqrt{2}}{8} & -\frac{2+\sqrt{-2+3\sqrt{3}}}{4\sqrt{3}}+i\frac{(\sqrt{3}+1)(1+2\sqrt{-2+3\sqrt{3}})}{4\sqrt{6}} \\ \frac{2-\sqrt{-2+3\sqrt{3}}}{4\sqrt{3}}+i\frac{(\sqrt{3}+1)(1-2\sqrt{-2+3\sqrt{3}})}{4\sqrt{6}} & -\frac{\sqrt{3}+1}{2}-\frac{\sqrt{-2+3\sqrt{3}}}{4}+i\frac{\sqrt{2}}{8} \end{pmatrix}$$

$$x_3 = \begin{pmatrix} \frac{\sqrt{3}+1}{2}+\frac{\sqrt{-2+3\sqrt{3}}}{4}-i\frac{\sqrt{2}}{8} & \frac{2+\sqrt{-2+3\sqrt{3}}}{4\sqrt{3}}-i\frac{(\sqrt{3}+1)(1+2\sqrt{-2+3\sqrt{3}})}{4\sqrt{6}} \\ \frac{-2+\sqrt{-2+3\sqrt{3}}}{4\sqrt{3}}+i\frac{(\sqrt{3}+1)(1-2\sqrt{-2+3\sqrt{3}})}{4\sqrt{6}} & \frac{\sqrt{3}+1}{2}-\frac{\sqrt{-2+3\sqrt{3}}}{4}+i\frac{\sqrt{2}}{8} \end{pmatrix}$$

Proof. The tripus is $M_{3,1}$ in the notation of Paoluzzi-Zimmermann, corresponding to the subgroup of Γ_3 which is the kernel of the homomorphism $\Gamma_3 \to \mathbb{Z}/3\mathbb{Z} = \{0,1,2\}$ determined by $h \mapsto 1$, $x \mapsto 1$. This is generated by the x_i, and a computation yields the formulae above. □

Theorem 5. *O_4 is isometric to the convex core of $\mathbb{H}^3/\Gamma_4$, where $\Gamma_4 = \langle h, x \rangle <$ $\mathrm{PSL}_2(\mathbb{C})$, with h and x given by*

$$h = \rho_{\pi/2} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad x = \begin{pmatrix} \frac{\sqrt{3}+1}{4\sqrt{2}}+i\frac{\sqrt{2+3\sqrt{3}}}{4} & x_{12} \\ x_{21} & \frac{\sqrt{3}+1}{4\sqrt{2}}+i\frac{\sqrt{2+3\sqrt{3}}}{4} \end{pmatrix},$$

where

$$x_{12} = \frac{3(\sqrt{3}+1)+2\sqrt{2+3\sqrt{3}}}{4\sqrt{2}} - i\frac{\sqrt{3}+1+\sqrt{2+3\sqrt{3}}}{4}$$
$$x_{21} = \frac{-3(\sqrt{3}+1)+2\sqrt{2+3\sqrt{3}}}{4\sqrt{2}} + i\frac{-(\sqrt{3}+1)+\sqrt{2+3\sqrt{3}}}{4}$$

This Kleinian group has presentation $\Gamma_4 = \langle h, x \mid h^4 = (hxhx^{-2})^4 = 1 \rangle$.

Proof. The proof is analogous to the case $n = 3$. Here $\alpha = \beta = \pi/12$ (again, see

[42], p. 117). Then the other standard geometric data is

$$\cosh C = 7 + 4\sqrt{3} \qquad \cosh B = \frac{3+2\sqrt{3}}{6}$$

$$\cosh L = \frac{2+2\sqrt{3}}{\sqrt{3}}$$

$$\cosh A = \frac{\sqrt{3}+1}{2} \qquad \cosh M = \sqrt{2}(2+\sqrt{3}).$$

The relevant matrices are

$$h = \rho_{\pi/2} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \rho_{2\pi/3} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$\sigma_\beta = \sigma_{\pi/12} = \begin{pmatrix} \frac{\sqrt{4+\sqrt{2}+\sqrt{6}}}{2\sqrt{2}} & -i\frac{\sqrt{4-\sqrt{2}-\sqrt{6}}}{2\sqrt{2}} \\ -i\frac{\sqrt{4-\sqrt{2}-\sqrt{6}}}{2\sqrt{2}} & \frac{\sqrt{4+\sqrt{2}+\sqrt{6}}}{2\sqrt{2}} \end{pmatrix}$$

$$\tau_{-L} = \begin{pmatrix} \left(\frac{2+2\sqrt{3}}{\sqrt{3}} + \sqrt{\frac{13+8\sqrt{3}}{3}}\right)^{-1/2} & 0 \\ 0 & \left(\frac{2+2\sqrt{3}}{\sqrt{3}} + \sqrt{\frac{13+8\sqrt{3}}{3}}\right)^{1/2} \end{pmatrix}$$

$$\tau_M = \begin{pmatrix} \left(\sqrt{2}(2+\sqrt{3}) + \sqrt{13+8\sqrt{3}}\right)^{1/2} & 0 \\ 0 & \left(\sqrt{2}(2+\sqrt{3}) + \sqrt{13+8\sqrt{3}}\right)^{-1/2} \end{pmatrix}$$

It may be observed that $\sqrt{13+8\sqrt{3}} = (\sqrt{3}+1)\sqrt{\frac{2+3\sqrt{3}}{2}}$. Using this, we modify the descriptions above.

$$\tau_{-L} = \begin{pmatrix} \left(\left(\frac{\sqrt{3}+1}{\sqrt{6}}\right)(2\sqrt{2} + \sqrt{2+3\sqrt{3}})\right)^{-1/2} & 0 \\ 0 & \left(\left(\frac{\sqrt{3}+1}{\sqrt{6}}\right)(2\sqrt{2} + \sqrt{2+3\sqrt{3}})\right)^{1/2} \end{pmatrix}$$

$$\tau_M = \begin{pmatrix} \left(\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)(\sqrt{3}+1+\sqrt{2+3\sqrt{3}})\right)^{1/2} & 0 \\ 0 & \left(\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)(\sqrt{3}+1+\sqrt{2+3\sqrt{3}})\right)^{-1/2} \end{pmatrix}$$

The description of x now follows from a computation. □

2.2 Frigerio's orbifolds

In this section we consider a family $\{M'_g \,|\, g \geq 2\}$ of noncompact finite volume hyperbolic manifolds with compact totally geodesic boundary, whose hyperbolic structures were described by R. Frigerio in [17], and a family of orbifolds O'_g branched–covered by the M'_g. For each $g \geq 2$, M'_g is the complement of a neighborhood in S^3 of a graph with two components, one of which is an unknotted circle (see [17], Fig. 1). The other component of this graph is the core of a trivially embedded handlebody of genus g, and so as Frigerio observes, M'_g may also be regarded as the complement of a knot in a handlebody. For $g = 2$, this is the "Adams-Reid knot" considered in [6] (cf. [17], Fig. 3 and [6], Fig. 4(b)), which is a particularly interesting example as it is the only member of this family which is "arithmetic" in a certain sense.

Frigerio describes the interior of M'_g as obtained by identifying in pairs the sides of a double cone P_{g+1} with vertices removed (see [17], Fig. 5 and Prop. 8). For $i \in \{0, \ldots, 2g+1\}$, define x_i to be the side–pairing map described at the beginning of [17], §2.1, taking the face $v_1 p_i p_{i+1}$ to $v_2 p_{i+1} p_{i+2}$ (the way in which these faces are identified depends on the parity of i). The double cone (with certain vertices removed and certain vertices truncated) may be realized in hyperbolic space so that the x_i are realized by isometries, as $g+1$ copies of the polyhedron Q_{g+1} of Figure 2.4(b) below, lined up around the dashed edge (cf. [17], §2.4).

The triangular and quadrilateral sides of Q_{g+1} are external, and the remaining sides are internal. The four–valent vertex visible is ideal, and the dihedral

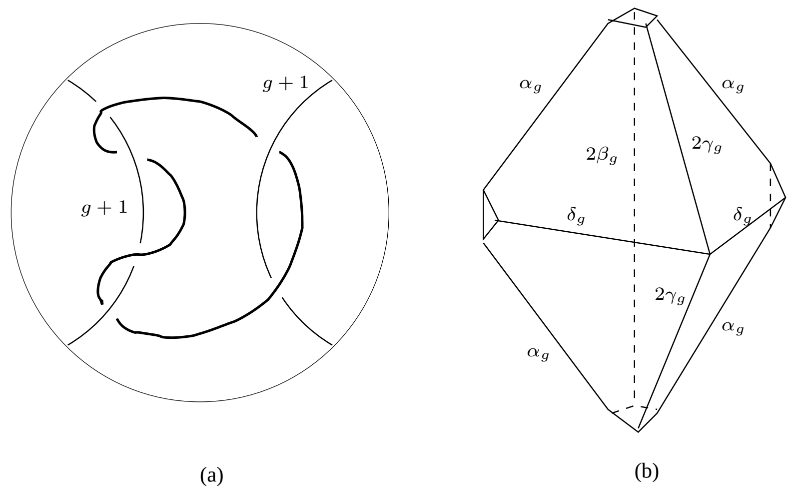


Figure 2.4: The orbifold O'_g

angles along the internal edges are determined by α_g, β_g, γ_g, δ_g satisfying

$$\alpha_g = \frac{\pi}{2g+2} \qquad \beta_g = 2\alpha_g \qquad \cos\gamma_g = \frac{1}{2\cos\alpha_g} \qquad \delta_g = \pi - 2\gamma_g$$

Summarizing the above, we have

Theorem (Frigerio)**.** *For each $g \geq 2$, the identification space of the geometric realization of P_{g+1} by the face–pairing isometries $\{x_i\}$ is a complete hyperbolic manifold with one totally geodesic boundary component and one cusp, homeomorphic to M'_g.*

The isometry group of M'_g is dihedral of order $2g+2$, the maximal cyclic subgroup of which is generated by a rotation r_g in the vertical edge of P_{g+1} (cf. [17], §3.3). Denote by O'_g the orbifold with boundary which is the quotient of M'_g by $\langle r_g \rangle$. Define an internal side–pairing of Q_{g+1} as follows: let ρ'_g be the rotation through the dashed edge in Figure 2.4(b) taking the right side containing it to the left, let λ'_g take the bottom left side containing the ideal vertex to its opposite, fixing the ideal vertex, and let μ'_g take the top left side containing the ideal vertex to its opposite, again fixing the ideal vertex. The following now follows from Frigerio's work and the Poincaré polyhedron theorem.

Corollary 2. *For each $g \geq 2$, the identification space of Q_{g+1} by the side–pairing described above is a complete hyperbolic orbifold with one geodesic boundary component and one cusp, homeomorphic to O'_g. This is the complement in the ball of the unknotted circle in Figure 2.4(a), with singular locus consisting of the labeled arcs.* $\Gamma'_g := \pi_1^{orb}(O'_g)$ *has presentation*

$$\Gamma'_g \cong \langle\, r, l, m \mid [l, m] = r^{g+1} = (rlrm)^{g+1} = 1\,\rangle.$$

The subgroup corresponding to M'_g is the kernel of a map onto $\mathbb{Z}/(g+1)\mathbb{Z} = \{0, 1, \dots, g\}$ given by $r \mapsto 1$, $l \mapsto 1$, $m \mapsto 0$, with generators given by

$$x_i = \begin{cases} h^{i+1}l^{-1}h^{-i} & i \text{ even} \\ h^i m h^{-i} & i \text{ odd} \end{cases}$$

The remainder of this section is devoted to describing an explicit representation of Γ'_g. As in the last section this requires an embedding of Q_{g+1}. An embedding of the expansion EQ_{g+1} is indicated in Figure 2.5.

Here the ideal vertex of Q_{g+1} has been placed at the point at infinity in the upper half space model, hence the faces of EQ_{g+1} with this vertex are contained in vertical half planes which form the rhombus pictured. The other two faces of EQ_{g+1} are contained in Euclidean hemispheres intersecting $\mathbb{C}\times\{0\}$ perpendicularly in the pictured circles. We require these hemispherical hyperplanes to meet in the geodesic between i and $-i$, and to be symmetric across the imaginary axis. This determines an embedding of EQ_{g+1} in $\mathbb{H}^3$ as the area above the two hyperplanes and inside of the rhombus. The bold arcs of the diagram, together with the vertices of the rhombus, are projections onto $\mathbb{C}$ of the edges of EQ_{g+1}, each of which contains an interior edge of Q_{g+1}, and the dihedral angles along these edges

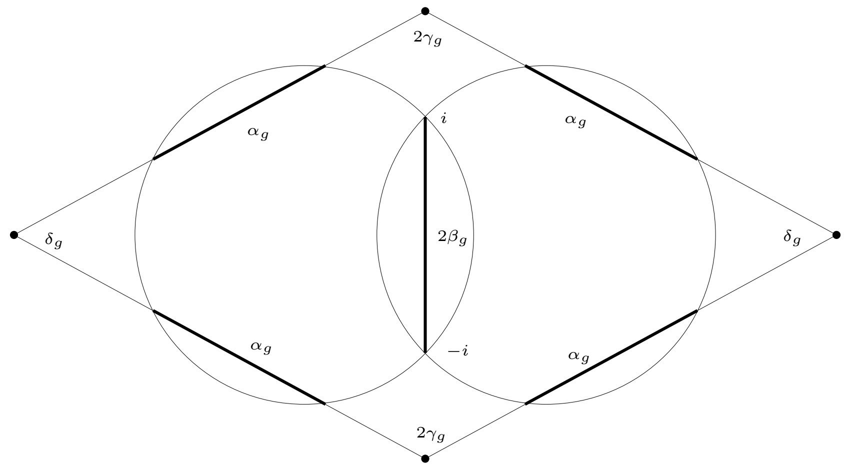


Figure 2.5: EQ_{g+1}

are indicated in the figure. Below we derive precise information in terms of these dihedral angles determining the radii and placement of the centers of the circles, and placement of the vertices.

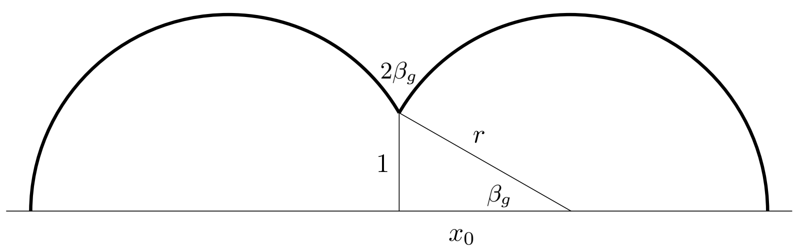


Figure 2.6: A cross section of Q_{g+1}

Figure 2.6 pictures the cross section of EQ_{g+1} by the plane over $\mathbb{R}$ in a neighborhood of the edge which is a geodesic from $-i$ to i. This edge passes through the plane over $\mathbb{R}$ at $(0,1)$, and so the pictured right triangle has height 1. The vertex opposite this edge is at the center of the circle defining the geodesic hyperplane containing the face above it, hence the (Euclidean) length r pictured is the radius of this circle and the center occurs at $(x_0, 0)$. Since the edge from i to $-i$ has dihedral angle 2β, and the faces which meet at this edge are exchanged by reflection in the hyperplane over the imaginary axis, it follows that the angle at the vertex of the triangle at $(0, x_0)$ is β_g, as indicated. Thus r and x_0 satisfy

$$r = \frac{1}{\sin \beta_g} \qquad\qquad x_0 = \frac{1}{\tan \beta_g}.$$

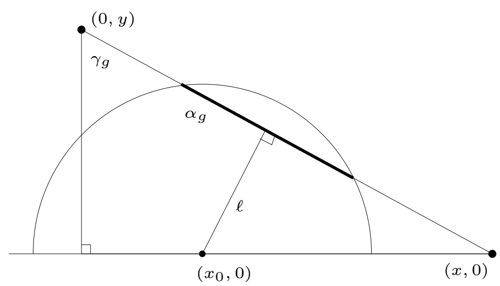


Figure 2.7: The upper right quarter of Q_{g+1}

Next we focus on the Euclidean geometry of the figures inscribed in the upper right hand quarter of Q_{g+1}, as shown in Figure 2.7. From the requirement that the bold edge of the diagram in Figure 2.7 have dihedral angle α_g, this description of ℓ follows

$$\ell = r \cos \alpha_g = \frac{\cos \alpha_g}{\sin \beta_g} = \frac{1}{2 \sin \alpha_g}.$$

Then using the fact that the triangle with side of length ℓ is similar to the larger one gives

$$\frac{\ell}{x - x_0} = \cos \gamma_g \qquad \Rightarrow \qquad x = \frac{\ell}{\cos \gamma_g} + x_0 = \cot \alpha_g + \cot \beta_g.$$

Furthermore, since $x/y = \tan \gamma_g$ we have

$$y = \frac{\cot \alpha_g + \cot \beta_g}{\tan \gamma_g}.$$

With the polyhedron embedded as in Figure 2.5, the side–pairings of faces containing the ideal vertex are realized by parabolic translations $\lambda_g'\colon\ z \mapsto z + (x +$

iy) and $\mu'_g\colon\ z \mapsto z+(x-iy)$ taking the lower left face of the rhombus to the upper right and the upper left to the lower right, respectively. The side–pairing of the remaining two faces is realized by the elliptic $\rho'_g = \rho_{2\beta_g}$ as in the previous section, since these faces intersect at angle 2β in the geodesic from i to $-i$. Assembling this information we have the following.

Theorem 6. *Frigerio's orbifold O'_g with totally geodesic boundary is isometric to the convex core of $\mathbb{H}^3/\Gamma'_g$, where $\Gamma'_g = \langle\, \lambda'_g, \mu'_g, \rho'_g \,\rangle < \mathrm{PSL}_2(\mathbb{C})$, given by*

$$\lambda'_g = \begin{pmatrix} 1 & x+iy \\ 0 & 1 \end{pmatrix} \qquad \mu'_g = \begin{pmatrix} 1 & x-iy \\ 0 & 1 \end{pmatrix} \qquad \rho'_g = \begin{pmatrix} \cos\beta_g & \sin\beta_g \\ -\sin\beta_g & \cos\beta_g \end{pmatrix}.$$

The numbers x and y above are given by

$$x = \frac{2\cos\beta_g + 1}{\sin\beta_g} \qquad\qquad y = \frac{\sqrt{2\cos\beta_g + 1}}{\sin\beta_g}.$$

Γ'_g has presentation $\langle\, r, l, m \mid [l,m] = r^{g+1} = (rlrm)^{g+1} = 1 \,\rangle$.

Proof. The simplification of the formulas for x and y follows from the relationships between α_g, β_g, and γ_g and a little trigonometry:

$$\cot\alpha_g = \frac{\cos(\beta_g/2)}{\sin(\beta_g/2)} = \sqrt{\frac{1+\cos\beta_g}{1-\cos\beta_g}} = \frac{1+\cos\beta_g}{\sin\beta_g}$$

Thus

$$x = \frac{1+\cos\beta_g}{\sin\beta_g} + \frac{\cos\beta_g}{\sin\beta_g} = \frac{1+2\cos\beta_g}{\sin\beta_g},$$

and the formula for y follows from the computation

$$\tan\gamma_g = \frac{\sin\gamma_g}{\cos\gamma_g} = 2\cos\alpha_g\sqrt{1-\frac{1}{4\cos^2\alpha_g}} = \sqrt{4\cos^2\alpha_g - 1} = \sqrt{2\cos\beta_g + 1}.$$

The group presentation and the fact that the quotient orbifold is homeomorphic to the labeled tangle of Figure 2.4(a) are the content of Corollary 2. □

2.3 Doubles and algebraic invariants

Finite volume hyperbolic manifolds with geodesic boundary share many of the rigidity properties of finite volume boundaryless hyperbolic manifolds. This is due to the fact that any orientable hyperbolic manifold with totally geodesic boundary M has associated a canonical boundaryless orientable hyperbolic manifold in which it embeds isometrically, its *double*

$$DM := M \cup_\partial \bar{M}.$$

Here $\bar{M}$ is a copy of M with the opposite orientation, and the identification between boundaries is induced by the identity map. DM inherits an atlas of charts from M and $\bar{M}$ defined as follows. Fix a geodesic hyperplane $H \subset \mathbb{H}^n$, and let r denote the reflection of $\mathbb{H}^n$ through H. Then $\bar{M}$ inherits an atlas of charts from M by composition with r. Namely, if $\phi\colon U \to \mathbb{H}^n$ is an orientation–preserving chart for M, then $r \circ \phi\colon \bar{U} \to \mathbb{H}^n$ is an orientation–preserving chart for $\bar{M}$.

A point contained in the interior of $M \hookrightarrow DM$ has a chart $\phi\colon U \to \mathbb{H}^n$ which maps homeomorphically onto an open subset of $\mathbb{H}^n$, where $U \subset (M - \partial M) \subset DM$. Similar charts exist for points of $\bar{M} \hookrightarrow DM$. If $x \in \partial M \hookrightarrow DM$, then x has a chart $\phi\colon U \to B_0$ to a half space bounded by a hyperplane H_0, where $\phi(\partial M \cap U) = \phi(U) \cap H_0$. Let r_0 be the reflection through H_0. A chart neighborhood of x in M is $U \cup \bar{U}$, with chart map given by ϕ on U and $r_0 \circ \phi$ on $\bar{U}$. Since $r_0 \circ \phi = (r_0 \circ r) \circ (r \circ \phi)$ is the composition of an orientation–preserving isometry with the chart map $r \circ \phi$ for $\bar{U}$, such charts satisfy the OP isometric overlaps condition.

Recall that to a complete hyperbolic manifold M with totally geodesic boundary is associated a developing map $\mathcal{D}\colon \widetilde{M} \to \mathbb{H}^n$ and an associated faithful, $\mathcal{D}$–equivariant holonomy representation $\mathcal{H}\colon\ \pi_1 M \to \mathrm{Isom}(\mathbb{H}^n)$ onto a Kleinian group Γ so that $\mathcal{D}$ induces an isometry from M to the convex core of $\mathbb{H}^n/\Gamma$ (Theorem 1 above). In order to relate the doubling process to this picture, we introduce some notation. Let $\{F_i\}$ be the collection of boundary components of M, and for each i fix a component H_i of the convex hull boundary for Γ which maps to the convex core boundary component corresponding to F_i, and let Λ_i be the stabilizer of H_i in Γ. Let r_i be reflection through H_i. For some fixed choice of r_0, let $\bar{\Gamma}$ be the conjugate of Γ by r_0, $\bar{\Gamma} = r_0 \Gamma r_0$.

Lemma 5. *DM is isometric to the convex core of*

$$D\Gamma = \langle\ \Gamma,\, \bar{\Gamma},\, \{r_0 r_i \,|\, i \neq 0\}\ \rangle,$$

by an isometry which restricts on M to the map induced by $\mathcal{D}$. As an abstract group, $D\Gamma$ is described as a sequence of HNN–extensions of a free product with amalgamation as $D\Gamma \cong \left(\Gamma *_{\Lambda_0} \bar{\Gamma}\right) *_{r_0 r_i}$

Proof. The description of $D\Gamma$ as an abstract group follows directly from the Klein-Maskit combination theorems (see [32]) and the choices of the r_i. The fact that DM is isometric to the convex core of $D\Gamma$ in the way described follows from observations of Morgan, see the discussion in [38] below Theorem 8.2. $\square$

When a Kleinian group Γ has finite covolume, Mostow rigidity asserts that any homeomorphism $\mathbb{H}^3/\Gamma \to \mathbb{H}^3/\Gamma'$ of the associated manifold to another is induced by conjugation; that is, there is an element $\gamma \in \mathrm{Isom}(\mathbb{H}^3)$ so that $\gamma\,\Gamma\,\gamma^{-1} = \Gamma'$, and the induced isometry of the quotient manifolds is homotopic to the original homeomorphism. The same statement holds for *geometrically finite* Kleinian groups—those whose convex core has finite volume—with totally geodesic convex core boundary, see [18] for a proof. Thus conjugacy invariants of such Kleinian groups are topological invariants of the associated finite volume hyperbolic manifolds with totally geodesic boundary.

One such invariant is the *trace field* of $\Gamma < \mathrm{PSL}_2(\mathbb{C})$, which is obtained by adjoining to $\mathbb{Q}$ the traces of elements of Γ. When Γ has finite covolume this is a number field—a finite extension of $\mathbb{Q}$. This follows from the Weil-Garland local rigidity theorem (see eg. §3.1 of [30] for an exposition). From the lemma above, it is clear that the trace field of a manifold with totally geodesic boundary is a subfield of the trace field of its double, hence the trace field of a finite volume hyperbolic manifold with totally geodesic boundary is also a number field.

Although the trace field of a finite-covolume Kleinian group is a topological invariant of the associated manifold, it is not necessarily true that the manifold's finite covers share its trace field (cf Example 3.3.1 of [30]). Since many questions about three-manifolds concern their behavior up to taking finite covers, it is useful to have *commensurability* invariants as well. We say $M = \mathbb{H}^3/\Gamma$ and $M' = \mathbb{H}^3/\Gamma'$ are commensurable if they share a finite cover; if M and M' have finite volume, this is equivalent to Γ having a finite–index subgroup conjugate to a subgroup of Γ'. A commensurability invariant for Γ may be obtained by taking the trace field of the finite–index subgroup $\Gamma^{(2)} = \langle \gamma^2 \mid \gamma \in \Gamma \rangle$. This is called the *invariant trace field* and denoted $k\Gamma$, see Theorem 3.3.4 of [30] for a proof that it is a commensurability invariant.

Another commensurability invariant of a Kleinian group Γ is the integrality of its traces. If the trace of every element $\gamma \in \Gamma$ is an algebraic integer, we say Γ has *integral traces*, otherwise we say it has a *nonintegral trace*. Clearly, if Γ has integral traces then every subgroup does as well. On the other hand, the relation between the trace of $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ and its square is given by

$$\operatorname{tr}\gamma^2 = \operatorname{tr}^2\gamma - 2.$$

Thus the trace of γ is integral over $\mathbb{Q}(\operatorname{tr}\gamma^2)$ if and only if $\operatorname{tr}\gamma^2$ is an integer. This implies that a nonintegral trace will pass to a finite–index subgroup; hence as stated above integrality of traces is a commensurability invariant.

Theorem 7. *The orbifold groups* Γ_3 *and* Γ_4 *have integral traces. Their trace fields are* $\mathbb{Q}(i\sqrt{-4+6\sqrt{3}})$ *and* $\mathbb{Q}(\sqrt{2}, i\sqrt{4+6\sqrt{3}})$*, respectively. Their invariant trace fields are* $k\Gamma_3 = \mathbb{Q}(i\sqrt{-4+6\sqrt{3}})$ *and* $k\Gamma_4 = \mathbb{Q}(i\sqrt{4+6\sqrt{3}})$*.*

Proof. These are each two–generator groups, and so the traces of all elements are given by integral polynomials in a short list of elements. Indeed, if we let $\langle\, h_i, x_i \,\rangle = \Gamma_i$ be the standard generating set described above, then we must merely determine the traces of h_i, x_i, and $h_i x_i$ (cf. equation 3.25 of [30]). Thus the following elements generate the trace field of O_3.

$$\text{tr } h_3 = 1 \qquad \text{tr } x_3 = \frac{\sqrt{3}}{2} + i\frac{(\sqrt{3}+1)\sqrt{-4+6\sqrt{3}}}{4}$$
$$\text{tr } h_3 x_3 = -\frac{2+\sqrt{3}}{2} + i\frac{(\sqrt{3}+1)\sqrt{-4+6\sqrt{3}}}{4}$$

If we let $\alpha_3 = \text{tr } x_3$ and denote by $\bar{\alpha}_3$ its complex conjugate, we see that $\text{tr } h_3 x_3 = -1 - \bar{\alpha}_3$. We have $(x-\alpha_3)(x-\bar{\alpha}_3) = x^2 - \sqrt{3}x + 2 + \sqrt{3}$, thus α_3 and $\bar{\alpha}_3$ are integral over $\mathbb{Q}(\sqrt{3})$ and hence are algebraic integers. This establishes the same for $\text{tr } x_3$ and $\text{tr } h_3 x_3$. Furthermore, $\mathbb{Q}(\text{tr } x_3, \text{tr } h_3 x_3)$ is a degree 2 extension of $\mathbb{Q}(\sqrt{3})$ (it clearly contains this field and is a nontrivial extension since $\mathbb{Q}(\sqrt{3}) \subset \mathbb{R}$). Since $\mathbb{Q}(i\sqrt{-4+6\sqrt{3}})$ clearly contains the trace field and is itself a degree 2 extension of $\mathbb{Q}(\sqrt{3})$, this is the trace field of Γ_3.

Now considering O_4, the following elements generate its trace field.

$$\text{tr } h_4 = \sqrt{2} \qquad \text{tr } x_4 = \frac{\sqrt{3}+1}{2\sqrt{2}} + i\frac{\sqrt{4+6\sqrt{3}}}{2\sqrt{2}}$$
$$\text{tr } h_4 x_4 = -\frac{\sqrt{3}+1}{2} + i\frac{\sqrt{4+6\sqrt{3}}}{2}.$$

Thus the trace field must contain $\sqrt{2}$ on account of h_4, and taking $\alpha_4 = \text{tr } x_4$ and

$\bar{\alpha}_4$ its complex conjugate, we find that tr $h_4 x_4 = -\sqrt{2}\bar{\alpha}_4$. We have

$$(x-\alpha_4)(x-\bar{\alpha}_4) = x^2 - \frac{\sqrt{3}+1}{\sqrt{2}}x + \sqrt{3}+1.$$

This establishes α_4 and $\bar{\alpha}_4$ as integral over $\mathbb{Q}(\sqrt{2},\sqrt{3})$ and hence as algebraic integers. (It is easy to check that $\frac{\sqrt{3}+1}{\sqrt{2}}$ is in fact an integer of $\mathbb{Q}(\sqrt{2},\sqrt{3})$.) Thus tr x_4 and tr $h_4 x_4$ are integers as well. Now the field obtained by adjoining tr x_4, tr $h_4 x_4$ to $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\text{tr } h_4)$ contains $\sqrt{3} = \sqrt{2}\text{tr } x_4 - \text{tr } h_4 x_4 - 1$, and hence has $\mathbb{Q}(\sqrt{2},\sqrt{3})$ as a subfield. This is a nontrivial extension, since $\mathbb{Q}(\sqrt{2},\sqrt{3}) \subset \mathbb{R}$, and by the above it is degree 2; hence it is degree 8 over $\mathbb{Q}$. Now the extension $\mathbb{Q}(\sqrt{2}, i\sqrt{4+6\sqrt{3}})$ clearly contains the trace field and is degree at most 8 over $\mathbb{Q}$, since it can be easily established that $i\sqrt{4+6\sqrt{3}}$ satisfies a polynomial of degree 4 over $\mathbb{Q}$. Thus this is the trace field.

The *invariant* trace field of Γ_i is determined by adjoining to $\mathbb{Q}$ the elements $\text{tr}^2\, h_i$, $\text{tr}^2\, x_i$, and $\text{tr}\, h_i \,\text{tr}\, x_i \,\text{tr}\, h_i x_i$ (cf. Lemma 3.5.7 of [30]). Clearly, for each i the square of $\text{tr}\, h_i$ is in $\mathbb{Q}$. The other elements are recorded below.

$$\begin{aligned}
\text{tr}^2\, x_3 &= -\frac{1+2\sqrt{3}}{2} + i\frac{(3+\sqrt{3})\sqrt{-4+6\sqrt{3}}}{4} \\
\text{tr}\, h_3 \,\text{tr}\, x_3 \,\text{tr}\, h_3 x_3 &= -\frac{4+3\sqrt{3}}{4} - i\frac{(\sqrt{3}+1)\sqrt{-4+6\sqrt{3}}}{4} \\
\text{tr}^2\, x_4 &= -\frac{\sqrt{3}}{2} + i\frac{(\sqrt{3}+1)\sqrt{4+6\sqrt{3}}}{4} \\
\text{tr}\, h_4 \,\text{tr}\, x_4 \,\text{tr}\, h_4 x_4 &= -2(\sqrt{3}+1)
\end{aligned}$$

It can be checked using this data that the invariant trace fields are as described above. □

We remark that $k\Gamma_3$ and $k\Gamma_4$ are the two degree–four number fields of discriminant -3312; this seems merely a somewhat bizarre coincidence. One of these, $k\Gamma_4$, arises as the invariant trace field of a manifold in the census of closed manifolds whose arithmetic data is computed with Snap [15]; the other does not. Thus doubling O_3 across its boundary yields an orbifold with invariant trace field $k\Gamma_3$ (see [40]), with a manifold cover that does not cover any closed manifold in the census. In fact, one can check using another arithmetic invariant, the *invariant quaternion algebra*, that the double of O_4 is also not commensurable with any manifold in the census.

Theorem 8. *The orbifold group Γ'_g has integral traces, and its trace field and invariant trace field are equal and equal to*

$$\mathbb{Q}\left(i\sqrt{2\cos\beta_g + 1}\right).$$

This has degree $\varphi(2g+2)$ over $\mathbb{Q}$, where φ is the Euler function.

Proof. The trace field of Γ'_g is generated as an extension of $\mathbb{Q}$ by adjoining traces of the elements λ'_g, μ'_g, ρ'_g, $\lambda'_g\mu'_g$, $\lambda'_g\rho'_g$, $\mu'_g\rho'_g$, and $\lambda'_g\mu'_g\rho'_g$ (see the discussion below Lemma 3.5.2 of [30]). Clearly the traces of λ'_g, μ'_g, and their product are all equal to 2, and the trace of ρ'_g is $2\cos\beta_g$. The traces of the remaining list elements are recorded below.

$$\mathrm{tr}\ \lambda'_g\rho'_g = 2\cos\beta_g - (x+iy)\sin\beta_g = -1 - i\sqrt{2\cos\beta_g + 1} \tag{2.5}$$

$$\mathrm{tr}\ \mu'_g\rho'_g = 2\cos\beta_g - (x-iy)\sin\beta_g = -1 + i\sqrt{2\cos\beta_g + 1} \tag{2.6}$$

$$\mathrm{tr}\ \lambda'_g\mu'_g\rho'_g = 2\cos\beta_g - 2x\sin\beta_g = -2 - 2\cos\beta_g \tag{2.7}$$

It is clear from the above that the trace field is $\mathbb{Q}(\cos\beta_g, i\sqrt{2\cos\beta_g+1})$. But $\cos\beta_g$ is contained in $\mathbb{Q}(i\sqrt{2\cos\beta_g+1})$, whence the description in the statement of the theorem.

Lemma 3.5.9 of [30] lists the various products of the traces above which generate the invariant trace field $k_{\Gamma'_g}$. In our case, the nontrivial examples are $\mathrm{tr}^2\rho'_g = \cos^2\beta_g$ and the product of tr ρ'_g with (1), (2), and (3) above. Adding the products of $2\cos\beta_g$ with (1) and (2) shows that the invariant trace field contains $\cos\beta_g$. It follows that one must adjoin $i\sqrt{2\cos\beta_g+1}$ as well to obtain the invariant trace field.

All traces of Γ_g' are obtained by applying polynomials with integer coefficients to the traces listed at the beginning of the proof, so in order to verify that Γ_g' has integral traces it suffices to check these. It is clear that $2\cos\beta_g$ is integral, since it is the sum of primitive $(2g+2)$nd roots of unity $\zeta_{2g+2}+\zeta_{2g+2}^{-1}$, each of which satisfies the (monic) $(2g+2)$nd cyclotomic polynomial. Thus tr ρ_g' and (3) above are each algebraic integers. The elements (1) and (2) above share a monic minimal polynomial over $\mathbb{Q}(\cos\beta_g)$,

$$p(x) = x^2 + 2x + 2\cos\beta_g + 2,$$

with integer coefficients. Hence they are algebraic integers as well. Thus Γ_g' has integer traces.

The minimal polynomial above displays the trace field as a degree 2 extension of $\mathbb{Q}(\cos\beta_g)$ (it is nontrivial since $\mathbb{Q}(\cos\beta_g) \subset \mathbb{R}$). But $\mathbb{Q}(\cos\beta_g)$ is also a degree 2 subfield of $\mathbb{Q}(\zeta_{2g+2}) = \mathbb{Q}(\cos\beta_g, i\sin\beta_g)$. Hence the trace field has the same degree over $\mathbb{Q}$ as $\mathbb{Q}(\zeta_{2g+2})$, which is well known to be $\varphi(2g+2)$. $\square$

The table below records the trace fields for low values of g. Note that in the case $g = 2$, the trace field is an imaginary quadratic extension of $\mathbb{Q}$. It follows easily from properties of φ that for all $g \geq 2$, the degree over $\mathbb{Q}$ of the trace field of Γ_g' is greater than two. However Γ_2' has similar algebraic properties to *arithmetic* hyperbolic 3-manifold groups. A finite volume noncompact hyperbolic 3-orbifold without boundary is arithmetic if and only if it has integral traces and its invariant trace field is an imaginary quadratic extension of $\mathbb{Q}$ (see Theorem 8.2.3 of [30]). One thus wonders whether O_2' isometrically embeds as a submanifold of an arithmetic hyperbolic 3-orbifold. Perhaps the most natural candidate for such an orbifold is the double DO'_2; however as we show below, this has a nonintegral trace. There is a *twisted double* of O'_2 which has integral traces, though, giving a positive answer to the above question. This is obtained by gluing O'_2 to $\bar{O}'_2$ by a nontrivial isometry of the boundary.

g	β_g	$\cos\beta_g$	$k\Gamma'_g$
2	$\pi/3$	$\frac{1}{2}$	$\mathbb{Q}\left(i\sqrt{2}\right)$
3	$\pi/4$	$\frac{\sqrt{2}}{2}$	$\mathbb{Q}\left(i\sqrt{1+\sqrt{2}}\right)$
4	$\pi/5$	$\frac{\sqrt{5}+1}{4}$	$\mathbb{Q}\left(i\frac{1+\sqrt{5}}{2}\right)$
5	$\pi/6$	$\frac{\sqrt{3}}{2}$	$\mathbb{Q}\left(i\sqrt{1+\sqrt{3}}\right)$

Table 2.1: Invariant trace fields of Γ'_g for low values of g.

In order to describe DO'_g algebraically, it is convenient to conjugate Γ'_g so that the doubling hyperplane is over $\mathbb{R}$. Then the reflection r_0 described above is realized by complex conjugation. Recall that Q_{g+1} is recovered from its expansion EQ_{g+1} by truncating open ends with geodesic hyperplanes perpendicular to sides of the expansion, yielding the four external faces of Figure 2.4(b). With EQ_{g+1} realized as in Figure 2.5, each of these truncating hyperplanes is a circle centered at one of the vertices of the rhombus. We choose to conjugate by an element which takes the circle centered at the bottom vertex in the figure to the hyperplane over $\mathbb{R}$; then the subgroup of Γ'_g stabilizing this hyperplane is conjugated into $\mathrm{PSL}_2(\mathbb{R})$. To identify the conjugating element, we collect one more piece of geometric information about Q_{g+1}. This is the length L of the labeled side of Figure 2.8.

Figure 2.8 is a picture of the cross–section of Q_{g+1} consisting of its intersection with the plane which is fixed by the reflection exchanging the two faces on either side of the dashed edge in Figure 2.4(b). With Q_{g+1} obtained by truncating EQ_{g+1} as embedded in Figure 2.5, the cross–section is by the vertical hyperplane over the imaginary axis. This cross–section is a hyperbolic pentagon with one ideal

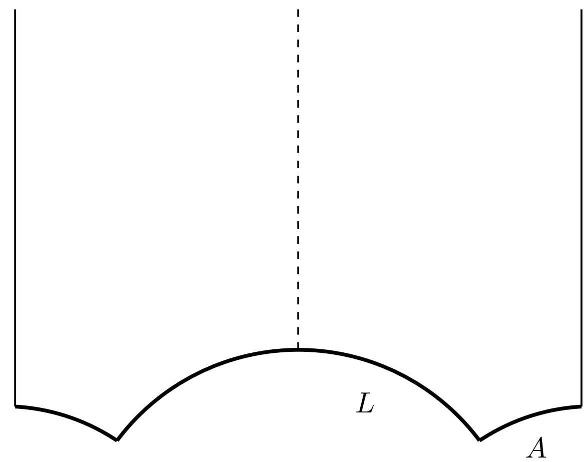


Figure 2.8: A vertical cross–section of Q_{g+1}

vertex and right angles at the other four. It has a reflective symmetry (which is the restriction of the other reflective symmetry of Q_{g+1}), whose axis is the dashed vertical line. With Q_{g+1} embedded as above, this vertical line is the intersection of the cross–section with the geodesic hyperplane over $\mathbb{R}$, and L is the distance between the hyperplane over $\mathbb{R}$ and the face containing the side labeled A in the figure. This side is contained in an external face of Q_{g+1}, and in this external face it is a side of a triangle with vertex angles α_g, β_g, and γ_g, opposite the vertex with angle α_g. Thus the hyperbolic law of cosines yields

$$\cosh A = \frac{\cos\beta_g \cos\gamma_g + \cos\alpha_g}{\sin\beta_g \sin\gamma_g} = \frac{\sqrt{2\cos\beta_g + 1}}{\sin\beta_g},$$

after trigonometric manipulation as before. Standard hyperbolic trigonometric identities for right–angled quadrilaterals show that $\sinh L\ \sinh A = 1$, yielding

$$\sinh L = \frac{\sin\beta_g}{\sqrt{2\cos\beta_g + \cos^2\beta_g}} \qquad \cosh L = \frac{\sqrt{2\cos\beta_g + 1}}{\sqrt{2\cos\beta_g + \cos^2\beta_g}}.$$

With the standard embedding of Q_{g+1}, the edge of length L is contained in the geodesic from $-i$ to i. The hyperbolic element which translates along this edge with translation length L is given by

$$\begin{pmatrix} \cosh(L/2) & i\sinh(L/2) \\ -i\sinh(L/2) & \cosh(L/2) \end{pmatrix}.$$

Define Γ_g to be the conjugate of Γ'_g by this matrix. Then $\Gamma_g = \langle\, \lambda_g, \mu_g, \rho_g \,\rangle$, where each of these is the conjugate by the matrix above of its correspondent in Γ'_g, subject to corresponding relations. Since ρ'_g itself fixes the edge from $-i$ to i, it commutes with the conjugating element and $\rho_g = \rho'_g$. On the other hand, conjugating λ'_g and μ'_g yields

$$\lambda_g = \begin{pmatrix} 1+\frac{i}{2}(x+iy)\sinh L & \frac{1}{2}(\cosh L+1)(x+iy) \\ \frac{1}{2}(\cosh L-1)(x+iy) & 1-\frac{i}{2}(x+iy)\sinh L \end{pmatrix}$$
$$\mu_g = \begin{pmatrix} 1+\frac{i}{2}(x-iy)\sinh L & \frac{1}{2}(\cosh L+1)(x-iy) \\ \frac{1}{2}(\cosh L-1)(x-iy) & 1-\frac{i}{2}(x-iy)\sinh L \end{pmatrix},$$

where x and y are as described earlier; namely

$$x = \frac{2\cos\beta_g + 1}{\sin\beta_g} \qquad\qquad y = \frac{\sqrt{2\cos\beta_g + 1}}{\sin\beta_g}.$$

By construction, the geodesic hyperplane over $\mathbb{R}$ is a convex hull boundary component for Γ_g; thus by Lemma 5, DO'_g is the quotient of $\mathbb{H}^3$ by $\langle\, \Gamma_g, \bar{\Gamma}_g \,\rangle$, where in this case $\bar{\Gamma}_g$ is actually obtained from Γ_g by complex conjugation. Armed with this description, we find a candidate element to display a nonintegral trace.

Lemma 6. *For each $g \geq 2$, the element $\lambda_g\mu_g^{-1}\bar{\mu}_g\bar{\lambda}_g^{-1}$ of $\langle\, \Gamma_g, \bar{\Gamma}_g \,\rangle$ has trace equal to*

$$\frac{4(2\cos\beta_g + 1)}{\cos\beta_g(2+\cos\beta_g)}.$$

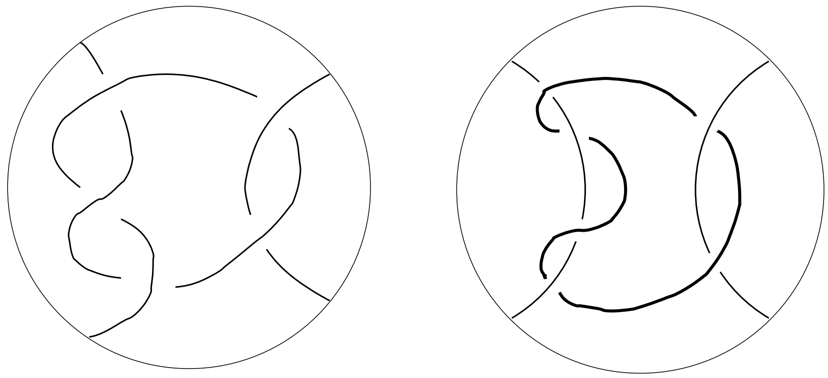
Proof. This is simply a computation. $\square$

In the case $g = 2$, this trace is $32/5$. Since the set of integers of $\mathbb{Q}(i\sqrt{2})$ is $\mathbb{Z}(i\sqrt{2})$, this is not an algebraic integer, and we have displayed a nonintegral trace for DO'_2. Thus DO'_2 is nonarithmetic.

2.4 Geometric limits

A convenient uniform way to consider the families of orbifolds in Sections 2.1 and 2.2 uses *orbifold Dehn surgery.* In the classical Dehn surgery construction, a solid torus is identified along its boundary with a torus $T \subset \partial M$ for some manifold M, so that the boundary of a meridional disk is identified with a specified *filling slope* γ. (A *slope* on ∂M is simply the isotopy class of a simple closed curve.) This yields a manifold $M(\gamma)$ with one fewer boundary component than M. Orbifold Dehn surgery is analogous, but instead of gluing a solid torus to M along T, one glues in an orbifold whose underlying topological space is a solid torus, with singular locus consisting of the core. This results in an orbifold we denote $O(n\gamma)$, where n is the order of the local group around points in the core, whose underlying topological space is $M(\gamma)$, but which has singular locus consisting of the core of the filling torus, with cone angle $2\pi/n$ there. If M has torus boundary components $T_1, \ldots, T_k$, surgeries on each of them yield an orbifold $O(n_1\gamma_1, n_2\gamma_2, \ldots, n_k\gamma_k)$.

In our situation, the orbifolds O_n of Section 2.1 are all obtained by an analogous orbifold Dehn surgery on the manifold O_∞ consisting of the complement in the ball of the tangle of Figure 2.9(a), by removing an open neighborhood of each tangle string and attaching to the resulting annulus a solid cylinder whose core has cone angle $2\pi/n$. Similarly, the orbifolds of Section 2.2 are obtained by orbifold Dehn surgery on the two arcs of the tangle complement O'_∞ of Figure 2.9(b). It



(a) (b)

Figure 2.9: These tangle complements are geometric limits for the O_n and O'_{g+1}.

may not be immediately clear how this is related to the Dehn surgery construction above — to see this, we note that doubling each of the tangles of Figure 2.9 across the boundary sphere yields a link in S^3, whose complement is DO_∞, respectively DO'_∞. Then O_n (resp. O'_g) is obtained by cutting $DO_\infty(n\gamma_1, n\gamma_2)$ (resp. $DO'_\infty((g+1)\gamma_1,(g+1)\gamma_2))$ in half, where γ_i is the slope determined by the intersection of the 4-punctured sphere with the appropriate torus boundary component.

Thurston's hyperbolic Dehn surgery theorem asserts that if M is a finite–volume hyperbolic 3-manifold with cusps $T_1, \ldots, T_k$, for (any) fixed choice of isomorphism $\mathrm{H}_1(\bigcup T_i) \to (\mathbb{Z}^2)^k$ there is a neighborhood U of $(\infty, \ldots, \infty)$ consisting entirely of hyperbolic surgeries. That is, if $(n_1\gamma_1, \ldots, n_k\gamma_k)$ has coordinates in U, the manifold obtained by truncating each T_i and surgering the resulting manifold along the slopes $n_i\gamma_i$ admits a hyperbolic structure. Furthermore, as the slopes approach $(\infty, \ldots, \infty)$, the surgered manifolds geometrically resemble M more and more closely, see [50] Ch. 5 or [10] Ch. 5. Conversely, a theorem of Kerckhoff asserts that if M is obtained from a hyperbolic manifold by drilling out a geodesic, M admits a hyperbolic structure in which a neighborhood of this geodesic is a cusp (cf. [24], [8]). Together, these theorems imply in our case that O_∞ (respectively,

O'_∞) admits a hyperbolic metric with totally geodesic boundary and annular cusps in neighborhoods of the tangle strings, and that this structure is a *geometric limit* of the structures on the O_n (resp. O'_{g+1}). (For background on geometric limits, see eg. [50] or [14], Ch. 6.)

The hyperbolic structure on O_∞ with totally geodesic boundary is examined at some length in a forthcoming paper of E. Chesebro and the author, so here we merely record the relevant theorem and refer the curious reader to [13]. We remark that this seems well known in the 3-manifolds community.

Theorem 9. *The manifold O_∞ admits a hyperbolic structure with totally geodesic boundary and tangle strings as rank one cusps homeomorphic to the convex core of $\mathbb{H}^3/\Gamma_\infty$, where $\Gamma_\infty = \langle r, s \rangle$ is generated by side–pairing isometries for a certain identification space of the regular ideal octahedron. Fixing an embedding of the ideal octahedron, these are realized by*

$$r = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \qquad\qquad s = \begin{pmatrix} 2i & 2-i \\ i & 1-i \end{pmatrix}$$

Γ_∞ is free on r and s. The totally geodesic 4–punctured sphere on the boundary of O_∞ is the quotient of the geodesic hyperplane over $\mathbb{R}$ by the group generated by the following three elements.

$$r^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad rsrs^{-2} = \begin{pmatrix} -1 & 5 \\ 0 & -1 \end{pmatrix} \qquad (srs)r^{-1}(srs)^{-1} = \begin{pmatrix} -14 & 25 \\ -9 & 16 \end{pmatrix}$$

This boundary subgroup is free on the above generators.

We devote the remainder of this section to determining the hyperbolic structure on O'_∞ by direct consideration of the geometric limit of the O'_g, and then studying its double and algebraic invariants, as in the previous section.

Lemma 7. *The manifold O'_∞ is obtained by gluing the faces of a polyhedron P'_∞,*

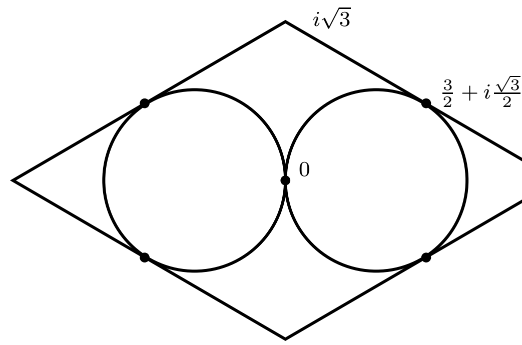


Figure 2.10: EP'_∞

whose expansion is pictured embedded in Figure 2.10, by side pairings λ', μ', *and* ρ' *given by*

$$\lambda' = \begin{pmatrix} 1 & 3+i\sqrt{3} \\ 0 & 1 \end{pmatrix} \qquad \mu' = \begin{pmatrix} 1 & 3-i\sqrt{3} \\ 0 & 1 \end{pmatrix} \qquad \rho' = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

The presentation for the group Γ'_∞ *generated by the side–pairings is* $\langle\, l, m, r \mid [l,m] = 1 \,\rangle$. *Let* $\gamma'_g = \left(\begin{smallmatrix} r & 0 \\ 0 & 1/r \end{smallmatrix}\right)$, *where* $r = \sqrt{\sin\beta_g}$. *Then the groups* $\gamma'_g\Gamma'_g(\gamma'_g)^{-1}$ *converge algebraically to* Γ'_∞, *and the orbifolds* O'_g *converge geometrically to* O_∞.

Proof. The action of λ' takes the face of EP'_∞ contained in the geodesic hyperplane over the line containing -3 and $-i\sqrt{3}$ to the one over the line between $i\sqrt{3}$ and 3. The other two vertical faces of EP'_∞ are exchanged by μ', and ρ' takes the face of EP'_∞ contained in the geodesic hyperplane over the right hand circle of Figure 2.10 to the face contained in the left hand circle. There is a visible correspondence between the sides of EP'_∞ and the sides of EQ_{g+1} as pictured in Figure 2.5. Indeed, truncating P'_∞ by a collection of open horoball neighborhoods of its ideal points equivariant under the side pairing maps, one obtains a polyhedron TP'_∞ combinatorially identical to that obtained by removing open neighborhoods of the internal edges of Q_{g+1} — segments of the bold edges in Figure 2.5 — equivariant under the action of its side pairing maps along with a horoball neighborhood of infinity.

Call the result of this truncation TQ_{g+1}. In both cases, the resulting polyhedron is homeomorphic to a ball, with new rectangular sides arising from boundaries of the truncating neighborhoods and internal sides giving rise to hexagons. Thus a homeomorphism may be constructed from the identification space of TP'_∞ to the identification space of TQ_{g+1} by the side–pairing maps induced by Γ'_{g+1}. But since the bold edges of Figure 2.5 comprise the singular locus in O'_{g+1}, the identification space of TQ_{g+1}, and hence that of TP'_∞, is homeomorphic to the complement of a regular neighborhood of the tangle of Figure 2.9(b). Filling most of the regular neighborhood back in with the horoballs, we find that O'_∞ is homeomorphic to the identification space of P'_∞ by the action of Γ.

The Poincare polyhedron theorem gives the stated presentation for Γ'_∞. From this it is easily seen that there is an epimorphism $\Gamma'_\infty \to \Gamma'_g$ given by $\lambda' \mapsto \lambda'_g$, $\mu' \mapsto \mu'_g$, $\rho' \mapsto \rho'_g$. Taking the conjugate of Γ'_g by γ'_g, we easily find that the images of λ'_g, μ'_g, and ρ'_g converge to λ', μ' and ρ', respectively, as $g \to \infty$. (This is due to the fact that $\beta_g = \pi/(g+1) \to 0$ and hence $\cos\beta_g \to 1$ as $g \to \infty$.) This proves algebraic convergence. The above discussion makes clear that the O'_{g+1} are obtained from O'_∞; thus geometric convergence follows from the hyperbolic Dehn surgery theorem. $\square$

A more favorable embedding of Γ'_∞ for studying DO'_∞ is its conjugate by

$$\begin{pmatrix} 1 & 0 \\ \frac{1+i\sqrt{3}}{2i\sqrt{3}} & 1 \end{pmatrix}.$$

This element moves the geodesic hyperplane through 0 and $\pm\frac{3}{2} - i\frac{\sqrt{3}}{2}$ to the geodesic hyperplane over $\mathbb{R}$, and the boundary subgroup of Γ'_∞ which fixes this hyperplane is conjugated by this element to a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. We record this in the lemma below. Re–embedding Γ'_∞ as its conjugate by the element above, we have the theorem below.

Theorem 10. *The manifold O'_∞ admits a hyperbolic structure with totally geodesic boundary and tangle strings as cusps isometric to the convex core of $\mathbb{H}^3/\Gamma'_\infty$, where $\Gamma'_\infty = \langle\, \rho, \lambda, \mu \,\rangle$ with $\rho = \rho'$ and λ' and μ' given by*

$$\lambda = \begin{pmatrix} -1 & 3+i\sqrt{3} \\ \frac{-1-i\sqrt{3}}{i\sqrt{3}} & 3 \end{pmatrix} \qquad \mu = \begin{pmatrix} i\sqrt{3} & 3-i\sqrt{3} \\ \frac{-2}{i\sqrt{3}} & 2-i\sqrt{3} \end{pmatrix}$$

The totally geodesic 4–punctured sphere on the boundary of O'_∞ is the quotient of the geodesic hyperplane over $\mathbb{R}$ by the group generated by the following three elements.

$$\rho = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \qquad \mu\rho\lambda = \begin{pmatrix} 1 & -6 \\ 1 & -5 \end{pmatrix} \qquad \mu\lambda^{-1}\rho\lambda\mu^{-1} = \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}$$

The totally geodesic surface subgroup is free on these generators.

Proof. The description of Γ'_∞ is easily verified by a computation, as is the description of the elements claimed to generate the boundary subgroup. To show that these do generate — and to explain their derivation — we refer to Figure 2.11.

This is a picture of the geodesic hyperplane through 0 and $\pm\frac{3}{2} - i\frac{\sqrt{3}}{2}$, and the images in it of the external sides of P'_∞ under certain elements of Γ'_∞. Recall that an external side of P'_∞ is obtained by truncating an open end of EP'_∞ by a geodesic hyperplane perpendicular to the sides bounding the end. There are four such sides in the case under consideration, which we denote by U, D, L, and R, standing for "up", "down", "left", and "right", respectively, according to their positions in Figure 2.10. Thus U is the face obtained by truncating the open quadrilateral end of EP'_∞ with sides intersecting S^2_∞ at 0, $i\sqrt{3}$, and $\pm\frac{3}{2} + i\frac{\sqrt{3}}{2}$, for example.

Since U, D, L and R are the only external sides of P'_∞, the boundary of the quotient manifold is obtained by pairing their sides, which is accomplished by elements of Γ'_∞. To keep track of this, we fix for reference the hyperplane containing

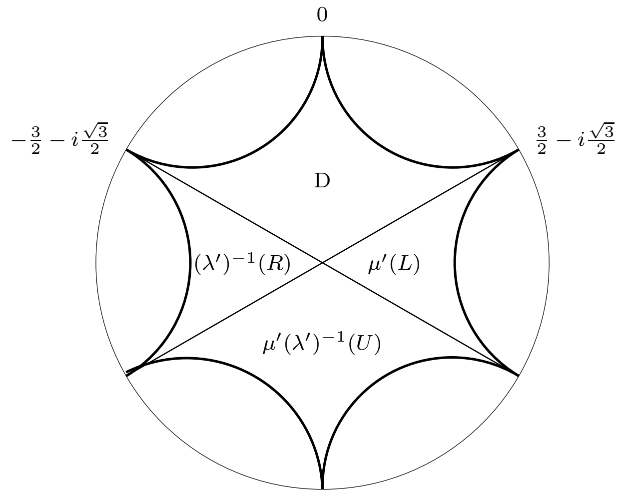


Figure 2.11: A fundamental domain for the action of a totally geodesic boundary subgroup of Γ'_∞.

D. Then $(\lambda')^{-1}$ and μ' move R and L, respectively, to abut D in this hyperplane as in the figure. Since $(\lambda')^{-1}$ moves U to abut L on its lower left side, $\mu'(\lambda')^{-1}$ moves U to abut $\mu(L)$ in the geodesic hyperplane containing D as pictured. But $\mu(\lambda')^{-1}(U) = (\lambda')^{-1}\mu(U)$ also abuts $(\lambda')^{-1}(R)$, since μ and λ commute, completing the description of the polygon in the figure. This is a fundamental domain for the action of the totally geodesic boundary subgroup of Γ'_∞ fixing this hyperplane, since its quotient is the entire boundary. Thus by the Poincaré polyhedron theorem applied to this ideal polygon, generators and relations may be obtained in terms of the side–pairing isometries.

It merely remains to find the side–pairing. We note that ρ' pairs the free sides of D, and also the sides of U which are not taken to abut the images of R or L by $\mu'(\lambda')^{-1}$. Hence $\mu'(\lambda')^{-1}\rho'\lambda'(\mu')^{-1}$ pairs the free sides of $\mu'(\lambda')^{-1}(U)$. Since ρ' also takes the right side of R to the left side of L, $\mu'\rho'\lambda'$ takes the corresponding side of $(\lambda')^{-1}(R)$ to that of $\mu'(L)$. Finally, we note that this group is free since the polygon is ideal. Conjugating by our special element yields the matrices of the statement of the lemma.

□

Corollary 3. *DO'_∞ is homeomorphic to the convex core of the Kleinian group $D\Gamma'_\infty = \langle\, \lambda, \mu, \rho', \bar{\lambda}, \bar{\mu} \,\rangle$, with presentation*

$$D\Gamma'_\infty \cong \langle\, l, m, r, \bar{l}, \bar{m} \mid [l,m] = [\bar{l}, \bar{m}] = 1,\ mrl = \bar{m}r\bar{l},\\ ml^{-1}rlm^{-1} = \bar{m}\bar{l}^{-1}r\bar{l}\bar{m}^{-1} \,\rangle.$$

Proof. This follows directly from the lemma above and Lemma 5. □

Corollary 4. *DO'_∞ is arithmetic, with trace field and invariant trace field equal to $\mathbb{Q}(i\sqrt{3})$. In particular, DO'_∞ has integral traces.*

Proof. The smallest field containing the entries of $D\Gamma'_\infty$ is clearly $\mathbb{Q}(i\sqrt{3})$. This must thus be equal to the trace field and invariant trace field, since they are subfields of the entry field and nontrivial extensions of $\mathbb{Q}$ (for DO'_∞ is a finite–volume hyperbolic 3-manifold).

To show that all traces of elements of $D\Gamma'_\infty$ are integral, we note that each generator is of the form

$$\begin{pmatrix} a & b(i\sqrt{3}) \\ \frac{c}{i\sqrt{3}} & d \end{pmatrix}$$

for $a, b, c, d \in \mathcal{O}_3$, where $\mathcal{O}_3$ is the ring of integers of $\mathbb{Q}(i\sqrt{3})$. The set of matrices of this form is a subgroup of $\mathrm{PSL}_2(\mathbb{Q}(i\sqrt{3}))$; hence $D\Gamma'_\infty$ is contained in this subgroup and thus has integral traces. This implies that DO'_∞ is arithmetic.

□

Chapter 3

Surfaces in knot complements

The material of this chapter has been previously published as [16].

3.1 Introduction

The presence of a totally geodesic surface in a hyperbolic 3–manifold has important topological implications. Long showed [27] that immersed totally geodesic surfaces lift to embedded nonseparating surfaces in finite covers, proving the virtual Haken and virtually positive β_1 conjectures for hyperbolic manifolds containing totally geodesic surfaces. Given this, it is natural to wonder about the extent to which topology constrains the existence of totally geodesic surfaces in hyperbolic 3–manifolds. Menasco–Reid have made the following conjecture [35]:

Conjecture (Menasco–Reid)**.** *No hyperbolic knot complement in* S^3 *contains a closed embedded totally geodesic surface.*

They proved this conjecture for alternating knots. The "Menasco–Reid" conjecture has been shown true for many other classes of knots, including almost alternating knots [5], Montesinos knots [41], toroidally alternating knots [3], 3–bridge and double torus knots [22], and knots of braid index 3 [28] and 4 [33]. For a knot in one of the above families, any closed essential surface in its complement has a topological feature which obstructs it from being even quasi-Fuchsian. In general, however, one cannot hope to find such obstructions. Adams–Reid have given examples of closed embedded quasi-Fuchsian surfaces in knot complements which volume calculations prove to be not totally geodesic [6].

On the other hand, C. Leininger has given evidence for a counterexample by constructing a sequence of hyperbolic knot complements in S^3 containing closed embedded surfaces whose principal curvatures approach 0 [26]. In this paper, we take an alternate approach to giving evidence for a counterexample, proving

Theorem 11. *There exist infinitely many hyperbolic knot complements in rational homology spheres containing closed embedded totally geodesic surfaces.*

This answers a question of Reid—recorded as Question 6.2 in [26]—giving counterexamples to the natural generalization of the Menasco–Reid conjecture to knot complements in rational homology spheres. Thus the conjecture, if true, must reflect a deeper topological feature of knot complements in S^3 than simply their rational homology.

Prior to proving Theorem 11, in Section 2 we prove

Theorem 12. *There exist infinitely many hyperbolic rational homology spheres containing closed embedded totally geodesic surfaces.*

This seems of interest in its own right, and the proof introduces many of the techniques we use in the proof of Theorem 11. Briefly, we find a two cusped hyperbolic manifold containing an embedded totally geodesic surface which remains totally geodesic under certain orbifold surgeries on its boundary slopes, and use the Alexander polynomial to show that branched covers of these surgeries have no rational homology.

In Section 3 we prove Theorem 11, giving examples using a similar branched covering construction. We construct a *three* cusped hyperbolic manifold N containing a totally geodesic surface intersecting two of the cusps, which remains totally geodesic in the orbifolds O_n resulting from n–fold orbifold filling on its boundary slopes, $n \geq 3$. We identify a slope on the third cusp, which does not intersect the totally geodesic surface, so that ordinary Dehn filling along this slope and the boundary slopes of the surface yields a rational homology sphere. Thus N may be regarded as a link complement in a rational homology sphere, and for odd $n \geq 3$ we apply a branched covering construction as above to find rational homology spheres containing one–cusped hyperbolic manifolds M_n covering the O_n as knot complements.

In the final section we determine the topology of the rational homology sphere S containing N as a link complement, and give some indication as to how similar techniques might be used to produce *integral* homology spheres containing closed embedded totally geodesic surfaces.

3.2 Theorem 12

Given a compact hyperbolic manifold M with totally geodesic boundary of genus g, gluing it to its mirror image $\bar{M}$ along the boundary yields a closed manifold DM—the "double" of M—in which the former ∂M becomes an embedded totally geodesic surface. One limitation of this construction is that this surface contributes half of its first homology to the first homology of DM, so that $\beta_1(DM) \geq g$. This is well known, but we include an argument to motivate our approach. Consider the relevant portion of the rational homology Mayer–Vietoris sequence for DM:

$$\cdots \to \mathrm{H}_1(\partial M, \mathbb{Q}) \xrightarrow{(i_*, -j_*)} \mathrm{H}_1(M, \mathbb{Q}) \oplus \mathrm{H}_1(\bar{M}, \mathbb{Q}) \to \mathrm{H}_1(DM, \mathbb{Q}) \to 0$$

The labeled maps i_* and j_* are the maps induced by inclusion of the surface into M and $\bar{M}$, respectively. Recall that by the "half lives, half dies" lemma (see eg. [20], Lemma 3.5), the dimension of the kernel of i_* is equal to g. Hence $\beta_1(M) \geq g$. Since the gluing isometry $\partial M \to \partial \bar{M}$ (the identity) extends over M, Ker i_* = Ker j_*, and so dim $\mathrm{Im}(i_*, -j_*) = g$. Hence

$$\mathrm{H}_1(DM, \mathbb{Q}) \cong \frac{\mathrm{H}_1(M, \mathbb{Q}) \oplus \mathrm{H}_1(\bar{M}, \mathbb{Q})}{\mathrm{Im}((i_*, -j_*))}$$

has dimension at least g.

Considering the above picture gives hope that by cutting DM along ∂M and regluing via some isometry $\phi\colon \partial M \to \partial M$ to produce a "twisted double" $D_\phi M$, one may reduce the homological contribution of ∂M. For then $j = i \circ \phi$, and if ϕ_* moves the kernel of the inclusion off of itself, then the argument above shows that the homology of $D_\phi M$ will be reduced. Below we apply this idea to a family of examples constructed by Zimmerman and Paoluzzi [42] which build on the "Tripus" example of Thurston [50].

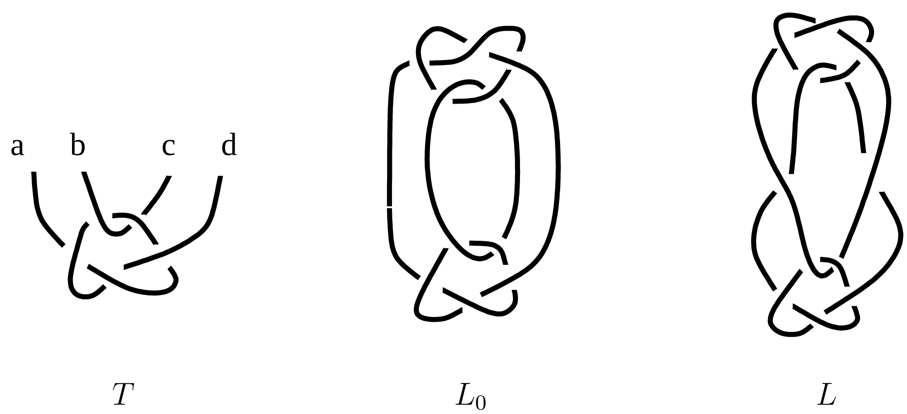


Figure 3.1: The tangle T and its double and twisted double.

The complement in the ball of the tangle T in Figure 3.1 is one of the minimal volume hyperbolic manifolds with totally geodesic boundary, obtained as an identification space of a regular ideal octahedron [37]. We will denote it O_∞. For $n \geq 3$, the orbifold O_n with totally geodesic boundary, consisting of the ball with cone locus T of cone angle $2\pi/n$, has been explicitly described by Zimmerman and Paoluzzi [42] as an identification space of a truncated tetrahedron. For each $k < n$ with $(k, n) = 1$, Zimmerman and Paoluzzi describe a hyperbolic manifold $M_{n,k}$ which is an n–fold branched cover of O_n. Topologically, $M_{n,k}$ is the n–fold branched cover of the ball, branched over T, obtained as the kernel of $\langle x, y\rangle = \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} = \langle t\rangle$ via $x \mapsto t$, $y \mapsto t^k$, where x and y are homology classes representing meridians of the two components of T.

We recall a well–known fact about isometries of spheres with 4 cone points:

Fact. *Let S be a hyperbolic sphere with 4 cone points of equal cone angle α, $0 \leq \alpha \leq 2\pi/3$, labeled a, b, c, d. Each of the following permutations of the cone points may be realized by an orientation-preserving isometry.*

$$(ab)(cd) \qquad\qquad (ac)(bd) \qquad\qquad (ad)(bc)$$

Using this fact, and abusing notation, let ϕ be the isometry $(ab)(cd)$ of ∂O_n, with labels as in Figure 3.1. Doubling the tangle ball produces the link L_0 in Figure 3.1, and cutting along the separating 4–punctured sphere and regluing via ϕ produces the link L, a *mutant* of L_0 in the classical terminology. Note that L and all of the orbifolds $D_\phi O_n$ contain the mutation sphere as a totally geodesic surface, by the fact above. ϕ lifts to an isometry $\tilde{\phi}$ of $\partial M_{n,k}$, and the twisted double $D_{\tilde{\phi}} M_{n,k}$ is the corresponding branched cover over L.

The homology of $D_{\tilde{\phi}} M_{n,k}$ can be described using the Alexander polynomial of L. The two variable Alexander polynomial of L is

$$\Delta_L(x, y) = \frac{1}{x^3}(x-1)(xy-1)(y-1)^2(x-y)$$

For the regular $\mathbb{Z}$-covering of $\mathrm{S}^3 - L$ given by $x \mapsto t^k$, $y \mapsto t$, the Alexander polynomial is

$$\Delta_L^k(t) = (t-1)\Delta(t^k, t) = \frac{1}{t^{3k-1}}(t-1)^5 \nu_{k-1}(t)\nu_k(t)\nu_{k+1}(t)$$

where $\nu_k(t) = t^{k-1} + t^{k-2} + \cdots + t + 1$. By a theorem originally due to Sumners [49] in the case of links, the first Betti number of $D_{\tilde{\phi}} M_{n,k}$ is the number of roots shared by $\Delta_L^k(t)$ and $\nu_n(t)$. Since this number is 0 for many n and k, we have a more precise version of Theorem 12. For example, we have

Theorem. *For $n > 3$ prime and $k \neq 0, 1, n-1$, $D_{\tilde{\phi}} M_{n,k}$ is a hyperbolic rational homology sphere containing an embedded totally geodesic surface.*

The techniques used above are obviously more generally applicable. Given any hyperbolic two–string tangle in a ball with totally geodesic boundary, one may double it to get a 2 component hyperbolic link in $\mathbb{S}^3$ and then mutate along the separating 4–punctured sphere by an isometry. By the hyperbolic Dehn surgery theorem and the fact above, for large enough n, $(n, 0)$ orbifold surgery on each component will yield a hyperbolic orbifold with a separating totally geodesic orbisurface. Then n–fold manifold branched covers can be constructed as above. One general observation about such covers follows from the following well-known fact, originally due to Conway:

Fact. *The one variable Alexander polynomial of a link is not altered by mutation; ie,*

$$\Delta_{L_0}(t, t) = \Delta_L(t, t)$$

when L is obtained from L_0 by mutation along a 4–punctured sphere.

In our situation, this implies the following:

Corollary. *A 2 component link in S^3 which is the twisted double of a tangle has no integral homology spheres among its abelian branched covers.*

Proof. A link L_0 which is the double of a tangle has Alexander polynomial 0. Therefore by the fact above,

$$\Delta_L^1(t) = (t-1)\Delta_L(t,t) = (t-1)\Delta_{L_0}(t,t) = 0$$

and so $D_{\tilde{\phi}}M_{n,1}$ has positive first Betti number by Sumners' theorem. The canonical abelian n^2–fold branched cover of L covers $D_{\tilde{\phi}}M_{n,1}$ and so also has positive first Betti number. Since the other n–fold branched covers of L have n–torsion, no branched covers of L have trivial first homology.

□

3.3 Theorem 11

In this section we construct hyperbolic knot complements in rational homology spheres containing closed embedded totally geodesic surfaces. The following "commutative diagram" introduces the objects involved in the construction and the relationships between them.

$$\begin{tikzcd}
N_n \arrow[r, "\text{Dehn}", "\text{filling}"'] \arrow[d] & M_n \arrow[r, "\text{Dehn}", "\text{filling}"'] \arrow[d] & S_n \\
N \arrow[r, "\text{orbifold}", "\text{filling}"'] & O_n &
\end{tikzcd}$$

Theorem 11 may now be more precisely stated as follows.

Theorem. *For each $n \geq 3$ odd, O_n is a one–cusped hyperbolic orbifold containing a totally geodesic sphere with four cone points of order n, M_n is a branched covering of O_n which is a one–cusped hyperbolic manifold, and S_n is a rational homology sphere.*

Before beginning the proof, we give a brief sketch of the strategy. We give an explicit polyhedral construction of a three–cusped hyperbolic manifold N containing an embedded totally geodesic 4–punctured sphere which intersects two of the cusps. For $n \geq 3$, we give the polyhedral decomposition of the orbifold O_n resulting from n–fold orbifold surgery on the boundary slopes of this 4–punctured sphere. From this it is evident that O_n is hyperbolic and the sphere remains totally geodesic. For odd $n \geq 3$, we prove that O_n has a certain one–cusped n–fold manifold cover M_n with a surgery S_n which is a rational homology sphere. This is accomplished by adapting an argument of Sakuma [47] to relate the homology of the n–fold cover $N_n \to N$ corresponding to $M_n \to O_n$, to the homology of S_n. M_n is thus a hyperbolic knot complement in a rational homology sphere, containing the closed embedded totally geodesic surface which is a branched covering of the totally geodesic sphere with 4 cone points in O_n.

Remark. It follows from the construction that the ambient rational homology sphere S_n covers an orbifold produced by n–fold orbifold surgery on each cusp of N. Thus by the hyperbolic Dehn surgery theorem, S_n is hyperbolic for $n >> 0$.

The proof occupies the remainder of the section. We first discuss the orbifolds O_n. For each n, O_n decomposes into the two polyhedra in Figure 3.2. Realized as a hyperbolic polyhedron, $P_a^{(n)}$ is composed of two truncated tetrahedra, each of which has two opposite edges of dihedral angle $\pi/2$ and all other dihedral angles $\pi/2n$, glued along a face. This decomposition is indicated in Figure 3.2 by the lighter dashed and dotted lines. The polyhedron $P_b^{(n)}$ has all edges with dihedral angle $\pi/2$ except for those labeled otherwise, and realized as a hyperbolic polyhedron it has all combinatorial symmetries and all circled vertices at infinity. By Andreev's theorem, polyhedra with the desired properties exist in hyperbolic space. Certain face pairings (described below) of $P_a^{(n)}$ yield a compact hyperbolic orbifold with totally geodesic boundary a sphere with 4 cone points of cone angle $2\pi/n$. Faces of $P_b^{(n)}$ may be glued to give a one–cusped hyperbolic orbifold with a torus cusp and totally geodesic boundary isometric to the boundary of the gluing of $P_a^{(n)}$. O_n is formed by gluing these orbifolds along their boundaries.

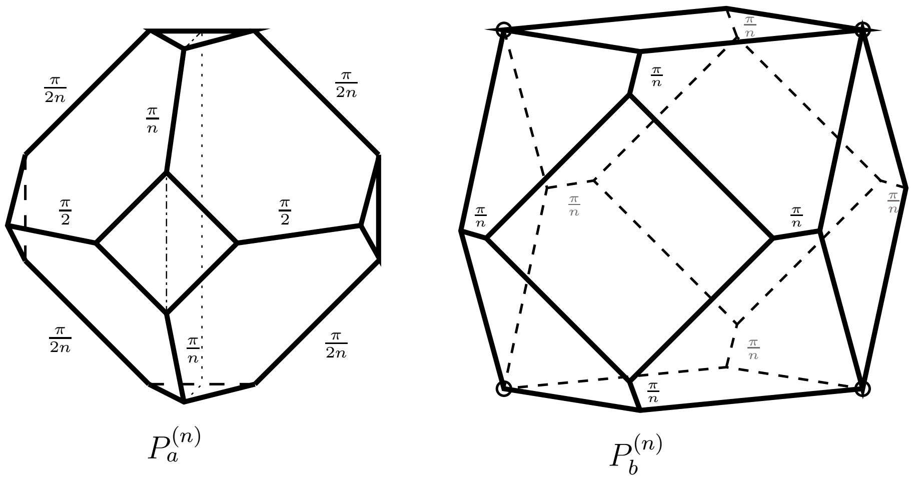


Figure 3.2: Cells for O_n

The geometric limit of the O_n as $n \to \infty$ is N, a 3–cusped manifold which decomposes into the two polyhedra in Figure 3.3. As above, realized as a convex polyhedron in hyperbolic space Q_a has all circled vertices at infinity. The edge of Q_a connecting face A to face C is finite length, as is the corresponding edge on the opposite vertex of A; all others are ideal or half-ideal, and all have dihedral angle $\pi/2$. Q_a has a reflective involution of order 2 corresponding to the involution of $P_a^{(n)}$ interchanging the two truncated tetrahedra. The fixed set of this involution on the back face is shown as a dotted line, and notationally we regard Q_a as having an edge there with dihedral angle π, splitting the back face into two faces X_5 and X_6. Q_b is the regular all-right hyperbolic ideal cuboctahedron.

Another remark on notation: the face opposite a face labeled with only a letter should be interpreted as being labeled with that letter "prime". For instance, the leftmost triangular face of Q_a has label C'. Also, each "back" triangular face of Q_b takes the label of the face with which it shares a vertex. For example, the lower left back triangular face is Y_1'.

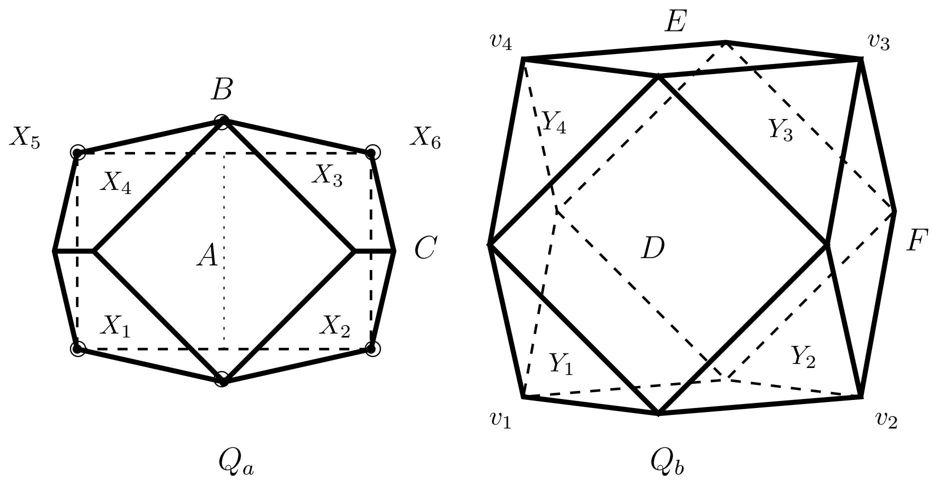


Figure 3.3: Cells for N

We first consider face pairings of Q_a producing a manifold N_a with two annulus cusps and totally geodesic boundary. Let r, s, and t be isometries realizing face pairings $X_1 \mapsto X_3$, $X_6 \mapsto X_4$, and $X_2 \mapsto X_5$, respectively. Poincaré's polyhedron theorem yields a presentation

$$\langle\, r, s, t \mid rst = 1 \,\rangle$$

for the group generated by r, s, and t. Note that this group is free on two generators, say s and t, where by the relation $r = t^{-1}s^{-1}$. Choose as the "boundary subgroup" (among all possible conjugates) the subgroup fixing the hyperbolic plane through the face A. A fundamental polyhedron for this group and its face–pairing isometries are in Figure 3.4. Note that the boundary is a 4–punctured sphere, and two of the three generators listed are the parabolics $t^{-1}s^{-1}ts^{-1}$ and $sts^{-1}t$, which generate the two annulus cusp subgroups of $\langle\, s, t \,\rangle$.

We now turn our attention to Q_b and the 3–cusped quotient manifold N_b. For $i \in \{1, 2, 3, 4\}$, let f_i be the isometry pairing the face $Y_i \to Y'_{i+1}$ so that $v_i \mapsto v_{i+1}$. Let g_1 be the *hyperbolic* isometry (that is, without twisting) sending

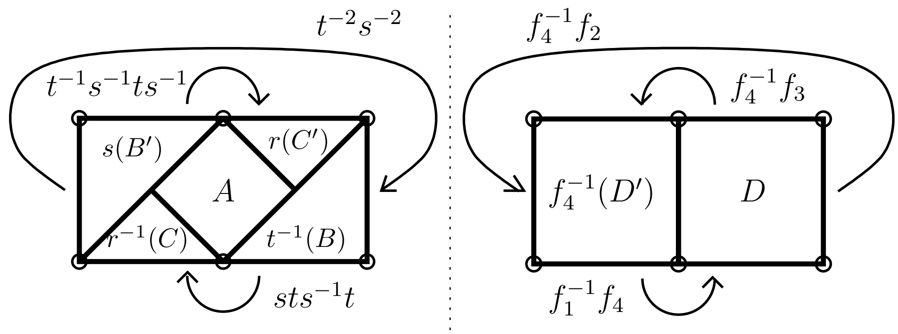


Figure 3.4: Totally Geodesic Faces of N_a and N_b

$E \to E'$, and g_2 the hyperbolic isometry sending $F \to F'$. The polyhedron theorem gives presentation

$$\begin{aligned}\langle f_1, f_2, f_3, f_4, g_1, g_2 \quad | \quad & f_1 g_2 f_2^{-1} g_1^{-1} = 1, \\ & f_2^{-1} g_2^{-1} f_3 g_1^{-1} = 1, \\ & f_3 g_2^{-1} f_4^{-1} g_1 = 1, \\ & f_4^{-1} g_2 f_1 g_1 = 1 \rangle\end{aligned}$$

for the group generated by the face pairings. The first 3 generators and relations may be eliminated from this presentation using Nielsen–Schreier transformations, yielding a presentation

$$\langle\ f_4, g_1, g_2 \mid f_4^{-1}[g_2, g_1] f_4 [g_2, g_1^{-1}] = 1\ \rangle$$

(our commutator convention is $[x, y] = xyx^{-1}y^{-1}$), where the first 3 relations yield

$$f_1 = g_1 g_2^{-1} g_1^{-1} f_4 g_2 g_1^{-1} g_2^{-1},\ f_2 = g_2^{-1} g_1^{-1} f_4 g_2 g_1^{-1},\ f_3 = g_1^{-1} f_4 g_2$$

The second presentation makes clear that the homology of N_b is free of rank 3, since each generator has exponent sum 0 in the relation. Faces D and D' make up the totally geodesic boundary of N_b. In Figure 3.4 is a fundamental polyhedron for the boundary subgroup fixing D, together with the face pairings generating the boundary subgroup.

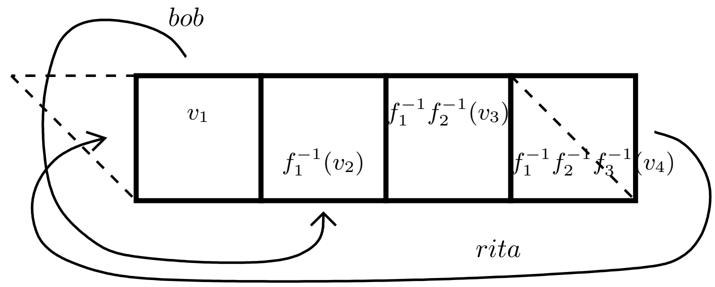


Figure 3.5: Closed Cusp of N_b

N_b has two annulus cusps, each with two boundary components on the totally geodesic boundary, and one torus cusp. A fundamental domain for the torus cusp in the horosphere centered at v_1 is shown in Figure 3.5, together with face pairing isometries generating the rank 2 parabolic subgroup fixing v_1. The generators shown are

$$bob = (f_4 g_1^{-1})^2 f_4 g_2 g_1^{-1} g_2^{-1},\ rita = (f_4 g_1^{-1})^3 f_4 g_2 g_1^{-1} g_2^{-1}$$

Note that $(bob)^{-4}(rita)^3$ is trivial in homology. This and $rita \cdot (bob)^{-1} = f_4 g_1^{-1}$ together generate the cusp subgroup fixing v_1. For later convenience, we now switch to the conjugate of this subgroup by f_4^{-1}, fixing v_4, and refer to $m = f_4^{-1}(f_4 g_1^{-1}) f_4 = g_1^{-1} f_4$ and $l = f_4^{-1}((bob)^{-4}(rita)^3) f_4$ as a "meridian-longitude" generating set for the closed cusp of N_b.

The totally geodesic 4–punctured spheres on the boundaries of N_a and N_b are each the double of a regular ideal rectangle, and we construct N by gluing N_a to N_b along them. Let us therefore assume that the polyhedra in Figure 3.3 are realized in hyperbolic space in such a way that face A of Q_a and face D of Q_b are in the same hyperbolic plane, with Q_a and Q_b in opposite half-spaces. Further arrange so that the polyhedra are aligned in the way suggested by folding the page containing Figure 3.4 along the dotted line down the center of the figure. With this arrangment, Maskit's combination theorem gives a presentation for the amalgamated group:

$$\begin{aligned}\langle\ f_4, g_1, g_2, s, t \ \ | \ \ & f_4^{-1}[g_2, g_1]f_4[g_2, g_1^{-1}] = 1,\\ & t^{-2}s^{-2} = f_4^{-1}g_2^{-1}g_1^{-1}f_4g_2g_1^{-1},\\ & sts^{-1}t = g_2g_1g_2^{-1}f_4^{-1}g_1g_2g_1^{-1}f_4,\\ & t^{-1}s^{-1}ts^{-1} = f_4^{-1}g_1^{-1}f_4g_2\ \rangle\end{aligned}$$

The first relation comes from N_b and the others come from setting the boundary face pairings equal to each other. Using Nielsen–Schreier transformations to eliminate g_2 and the last relation, the resulting presentation is

$$\begin{aligned}&\langle\ f_4, g_1, s, t \mid\\ &\quad f_4^{-2}g_1[f_4t^{-1}s^{-1}ts^{-1}, g_1]f_4[f_4t^{-1}s^{-1}ts^{-1}, g_1^{-1}]g_1^{-2}f_4g_1 = 1,\\ &\quad t^{-2}s^{-2} = f_4^{-1}st^{-1}stf_4^{-1}g_1^{-1}f_4^2t^{-1}s^{-1}ts^{-1}g_1^{-1},\\ &\quad sts^{-1}t = f_4^{-1}g_1f_4t^{-1}s^{-1}ts^{-1}g_1st^{-1}stf_4^{-2}g_1f_4t^{-1}s^{-1}ts^{-1}g_1^{-1}f_4\ \rangle\end{aligned}$$

Replace g_1 with the meridian generator of the closed cusp of N_b, $m = g_1^{-1}f_4$, and add generators $m_1 = f_4^{-1}mf_4$ and $m_2 = st^{-1}stm_1t^{-1}s^{-1}ts^{-1}$, each conjugate to m, yielding

$$\begin{aligned}&\langle\ f_4, m, m_1, m_2, s, t \mid m_1 = f_4^{-1}mf_4,\ m_2 = st^{-1}stm_1t^{-1}s^{-1}ts^{-1},\\ &\quad m_1^{-1}t^{-1}s^{-1}ts^{-1}f_4m^{-1}m_2mf_4^{-1}st^{-1}stm_1m^{-1} = 1,\\ &\quad s^2t^2f_4^{-1}m_2mf_4^{-1} = 1,\\ &\quad t^{-1}st^{-1}s^{-1}m^{-1}f_4t^{-1}s^{-1}ts^{-1}f_4m^{-1}m_2^{-1}m = 1\ \rangle\end{aligned}$$

Note that after abelianizing, each of the last two relations expresses $f_4^2 = m^2s^2t^2$, since m_1 and m_2 are conjugate to m and therefore identical in homology. In light of this, we replace f_4 by $u = t^{-1}s^{-1}f_4m^{-1}$, which has order 2 in homology. This yields

$$
\begin{aligned}
&\langle\, m, m_1, m_2, s, t, u \mid \\
&\quad m_1^{-1}m^{-1}u^{-1}t^{-1}s^{-1}mstum = 1 \qquad (3.1)\\
&\quad m_2^{-1}st^{-1}stm_1t^{-1}s^{-1}ts^{-1} = 1 \qquad (3.2)\\
&\quad m_1^{-1}t^{-1}s^{-1}t^2um_2u^{-1}t^{-2}stm_1m^{-1} = 1 \qquad (3.3)\\
&\quad s^2t^2m^{-1}u^{-1}t^{-1}s^{-1}m_2u^{-1}t^{-1}s^{-1} = 1 \qquad (3.4)\\
&\quad t^{-1}st^{-1}s^{-1}m^{-1}stumt^{-1}s^{-1}t^2um_2^{-1}m = 1\,\rangle \qquad (3.5)
\end{aligned}
$$

Let R_i denote the relation labeled (i) in the presentation above. In the abelianization, R_1 and R_2 set $m_1 = m$ and $m_2 = m_1$, respectively, R_3 disappears, and the last two relations set $u^2 = 1$. Therefore

$$\mathrm{H}_1(N) \cong \mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z} = \langle m\rangle \oplus \langle s\rangle \oplus \langle t\rangle \oplus \langle u\rangle$$

(We will generally blur the distinction between elements of π_1 and their homology classes.)

The boundary slopes of the totally geodesic 4–punctured sphere coming from ∂N_a and ∂N_b are represented in $\pi_1(N)$ by $t^{-1}s^{-1}ts^{-1}$ and $sts^{-1}t$. Let O_n be the finite volume hyperbolic orbifold produced by performing face identifications on $P_a^{(n)}$ and $P_b^{(n)}$ corresponding to those on Q_a and Q_b. O_n is geometrically produced by n–fold orbifold filling on each of the above boundary slopes of N. Appealing to the polyhedral decomposition, we see that the separating 4–punctured sphere remains totally geodesic, becoming a sphere with 4 cone points of order n. Our knots in rational homology spheres are certain manifold covers of the O_n. In order to understand the homology of these manifold covers, we compute the homology of the corresponding abelian covers of N.

Let $p\colon \widetilde{N} \to N$ be the maximal free abelian cover; that is, $\widetilde{N}$ is the cover corresponding to the kernel of the map $\pi_1(N) \to \mathrm{H}_1(N) \to \mathbb{Z}^3 = \langle x, y, z\rangle$ by

$$m \mapsto x \qquad s \mapsto y \qquad t \mapsto z \qquad u \mapsto 1$$

Let X be a standard presentation 2-complex for $\pi_1(N)$ and $\widetilde{X}$ the 2-complex covering X corresponding to $\widetilde{N} \to N$. Then the first homology and Alexander module of $\widetilde{X}$ are naturally isomorphic to those of $\widetilde{N}$, since N is homotopy equivalent to a cell complex obtained from X by adding cells of dimension 3 and above. The covering group $\mathbb{Z}^3$ acts freely on the chain complex of $\widetilde{X}$, so that it is a free $\mathbb{Z}[x, x^{-1}, y, y^{-1}, z, z^{-1}]$–module. Below we give a presentation matrix for the Alexander module of $\widetilde{X}$:

$$\begin{pmatrix}
\frac{1-yz+xyz}{x^2yz} & 0 & -1 & -\frac{y^2z^2}{x} & \frac{-1+yz+z^2}{xz^2} \\
-\frac{1}{x} & \frac{y^2}{x} & \frac{x-1}{x} & 0 & 0 \\
0 & -\frac{1}{x} & \frac{z}{xy} & \frac{yz}{x} & -\frac{1}{x} \\
\frac{x-1}{x^2yz} & -\frac{(x-1)(y+z)}{xz} & \frac{x-1}{xyz} & \frac{y(x-z)}{x} & \frac{1-2x+xz}{xz^2} \\
\frac{x-1}{x^2z} & -\frac{y(x-1)(y-1)}{xz} & \frac{(x-1)(-1+y-z)}{xyz} & \frac{y(-x+xy+xyz-yz)}{x} & \frac{x+y-2xy}{xz^2} \\
\frac{x-1}{x^2} & 0 & -\frac{z(x-1)}{xy} & -\frac{yz(x+yz)}{x} & \frac{y+xz}{xz}
\end{pmatrix}$$

The rows of the matrix above correspond to lifts of the generators for $\pi_1(N)$ sharing a basepoint, ordered as $\{\tilde{m}, \tilde{m}_1, \tilde{m}_2, \tilde{s}, \tilde{t}, \tilde{u}\}$ reading from the top down. These generate $\mathrm{C}_1(\widetilde{X})$ as a $\mathbb{Z}[x, x^{-1}, y, y^{-1}, z, z^{-1}]$–module. The columns are the Fox free derivatives of the relations in terms of the generators, giving a basis for the image of $\partial\mathrm{C}_2(\widetilde{X})$. For a generator g above, let p_g be the determinant of the square matrix obtained by deleting the row corresponding to $\tilde{g}$. These polynomials are

$$\begin{aligned}
p_m &= \tfrac{-1}{x^4z^2}(x-1)^2(y-1)(z-1)(y+z+4yz+y^2z+yz^2) \\
p_{m_1} &= \tfrac{1}{x^4z^2}(x-1)^2(y-1)(z-1)(y+z+4yz+y^2z+yz^2) \\
p_{m_2} &= \tfrac{-1}{x^4z^2}(x-1)^2(y-1)(z-1)(y+z+4yz+y^2z+yz^2) \\
p_s &= \tfrac{1}{x^4z^2}(x-1)(y-1)^2(z-1)(y+z+4yz+y^2z+yz^2) \\
p_t &= \tfrac{-1}{x^4z^2}(x-1)(y-1)(z-1)^2(y+z+4yz+y^2z+yz^2) \\
p_u &= 0
\end{aligned}$$

The Alexander polynomial of $\mathrm{H}_1(\widetilde{N})$ is the greatest common factor:

$$\Delta(x,y,z) = (x-1)(y-1)(z-1)(y+z+4yz+y^2z+yz^2)$$

up to multiplication by an invertible element of $\mathbb{Z}[x,x^{-1},y,y^{-1},z,z^{-1}]$.

Let N_∞ be the infinite cyclic cover of N factoring through $\widetilde{N}$ given by

$$m \mapsto x^2 \qquad s \mapsto x \qquad t \mapsto x \qquad u \mapsto 1$$

Then the chain complex of N_∞ is a Λ–module, where $\Lambda = \mathbb{Z}[x,x^{-1}]$, and specializing the above picture yields an Alexander polynomial

$$\begin{aligned}
\Delta_\infty(x) &= (x^2-1)(x-1)^2(2x+4x^2+2x^3) \\
&= 2x(x-1)^3(x+1)^3
\end{aligned}$$

Let N_n be the n–fold cyclic cover of N factoring through N_∞. For n odd, N_n has three cusps, since m, $sts^{-1}t$, and $t^{-1}s^{-1}ts^{-1}$ map to $x^{\pm 2}$, which generates $\mathbb{Z}/n\mathbb{Z}$. Let S_n be the closed manifold obtained by filling N_n along the slopes covering m, $sts^{-1}t$, and $t^{-1}s^{-1}ts^{-1}$. Theorem 11 follows quickly from the following lemma.

Lemma 8. *For odd $n \geq 3$, S_n is a rational homology sphere.*

Proof. The proof is adapted from an analogous proof of Sakuma concerning link complements in S^3 ([47], see also [21], §5.7). We begin by noting that the chain complex of N is naturally isomorphic to $\mathbb{Z} \otimes_\Lambda C_*(N_\infty)$, where $\mathbb{Z}$ is made a Λ–module by the augmentation map. This follows from the fact that N is the quotient of N_∞ by the cyclic group generated by the covering transformation corresponding to x. Similarly, the chain complex of N_n is isomorphic to $\Lambda/(x^n - 1) \otimes_\Lambda C_*(N_\infty)$.

Note that $x^n - 1 = (x-1)\nu_n$, where $\nu_n(x) = x^{n-1} + x^{n-2} + \ldots + x + 1$. The key observation is that there is a short exact sequence of coefficient modules

$$0 \to \mathbb{Z} \xrightarrow{\nu_n} \Lambda/(x^n - 1) \to \Lambda/(\nu_n) \to 0,$$

where the map on the left is multiplication by ν_n. Tensoring with $C_*(N_\infty)$ gives a short exact sequence of chain complexes (recall that $C_*(N_\infty)$ is a free Λ–module, and therefore flat), which gives rise to an exact sequence in homology

$$\ldots \ \mathrm{H}_1(N) \xrightarrow{tr} \mathrm{H}_1(N_n) \to \mathrm{H}_1(N; \Lambda/(\nu_n)) \to 0$$

where tr is the transfer map, $tr(h) = h + x.h + \cdots + x^{n-1}.h$ for a homology class h. That the final map above is trivial follows from the fact that the transfer map is injective on H_0. Another exact sequence of chain complexes allows a different description of $\mathrm{H}_1(N; \Lambda/(\nu_n))$:

$$0 \to C_*(N_\infty) \xrightarrow{\nu_n} C_*(N_\infty) \to \Lambda/(\nu_n) \otimes_\Lambda C_*(N_\infty) \to 0$$

gives rise to a homology exact sequence

$$\ldots \ \mathrm{H}_1(N_\infty) \xrightarrow{\nu_n} \mathrm{H}_1(N_\infty) \to \mathrm{H}_1(N; \Lambda/(\nu_n)) \to 0.$$

Again, triviality of the final map follows from the fact that multiplication by ν_n induces an injection on $\mathrm{H}_0(N_\infty)$. This describes $\mathrm{H}_1(N;\Lambda/(\nu_n)) = \mathrm{H}_1(N_\infty)/\nu_n\mathrm{H}_1(N_\infty)$. Since the Alexander polynomial of N_∞ does not share roots with ν_n, H_n is a torsion $\mathbb{Z}$–module; that is, a finite abelian group (see eg. Lemma 7.2.8 of [23]).

The lemma follows from a comparison between $\mathrm{H}_1(S_n)$ and $\mathrm{H}_1(N;\Lambda/(\nu_n))$. The Mayer-Vietoris sequence implies that $\mathrm{H}_1(S_n)$ is obtained as the quotient of $\mathrm{H}_1(N_n)$ by the subgroup generated by transfers of the meridians. If N were a link complement in S^3, it would immediately follow that $\mathrm{H}_1(N;\Lambda/(\nu_n)) = \mathrm{H}_1(S_n)$, since the homology of a link complement is generated by meridians. In our case we have

$$\mathrm{H}_1(N;\Lambda/(\nu_n)) = \mathrm{H}_1(N_n)/\langle\ tr(m), tr(s), tr(t), tr(u)\ \rangle,$$

whereas

$$\mathrm{H}_1(S_n) = \mathrm{H}_1(N_n)/\langle\ tr(m), tr(2s), tr(2t)\ \rangle.$$

However one observes that $\mathrm{H}_1(S_n) \to \mathrm{H}_1(N;\Lambda/(\nu_n))$ is an extension of degree at most 8 (since u has order 2 in $\mathrm{H}_1(N)$), and so $\mathrm{H}_1(S_n)$ is also finite.

□

Let M_n be the manifold obtained by filling two of the three cusps of N_n along the slopes covering $sts^{-1}t$ and $t^{-1}s^{-1}ts^{-1}$. M_n is a branched cover of O_n, which we have geometrically described as produced by n–fold orbifold filling along $sts^{-1}t$ and $t^{-1}s^{-1}ts^{-1}$. M_n contains a closed totally geodesic surface covering the totally geodesic sphere with 4 cone points in O_n. S_n is produced by filling the remaining cusp of M_n along the meridian covering m to give a closed manifold. Since S_n is a rational homology sphere, M_n is a knot complement in a rational homology sphere, and we have proven Theorem 11.

3.4 Fillings

Performing ordinary Dehn filling along the 3 meridians of N specified in the previous section yields a manifold S, which is easily seen to be the connected sum of two spherical manifolds. The half arising from the truncated tetrahedra is the quotient of S^3, regarded as the set of unit quaternions, by the subgroup $\langle i, j, k \rangle$. The half arising from the cuboctahedron is the lens space $\mathrm{L}(4,1)$. The manifolds S_n may be regarded as n–fold branched covers over the 3 component link L in S consisting of the cores of the filling tori.

Since the meridians $t^{-1}s^{-1}ts^{-1}$ and $sts^{-1}t$ represent squares of primitive elements in the homology of N, any cover of S branched over L will have nontrivial homology of order 2 coming from the transfers of s and t. However, it is possible that techniques similar to those above may be used to create knot complements in integral homology spheres. If the manifold N above—in addition to its geometric properties—had trivial nonperipheral *integral* homology, then S would be an integral homology sphere. Porti [43] has supplied a formula in terms of the Alexander polynomial for the order of the homology of a cover of an integral homology sphere branched over a link, generalizing work of Mayberry–Murasugi in the case of S^3 [34]. Using this formula, the order of the homology of branched covers of S could be easily checked.

Chapter 4

Further directions

4.1 Menasco-Reid

The Menasco-Reid conjecture itself remains open, but even beyond this very little seems known about the subject of totally geodesic surfaces embedded or immersed in hyperbolic knot complements in S^3. The work of Adams et al ([1], [4]) has demonstrated the existence of totally geodesic *Seifert surfaces* (that is, orientable surfaces with a single longitudinal boundary component) in certain hyperbolic knot complements, for instance the (n, n, n) pretzel knots. However, we do not know of any other examples of embedded totally geodesic surfaces in hyperbolic knot complements. In particular, the following more general question remains unanswered.

Question. *Does there exist a knot complement containing a separating totally geodesic surface?*

The "co–Menasco-Reid" conjecture below addresses a different special case of this question, which is of particular interest to us in light of examples herein.

Conjecture. *No hyperbolic knot complement in S^3 contains a totally geodesic*

Conway sphere (that is, a separating four punctured sphere with meridional boundary).

There are many knots with crossing number not much greater than 10 on which this conjecture may be easily tested using volume calculations with Snappea. The candidate with lowest crossing number is the knot 10_{153} in the standard notation; this possesses an incompressible Conway sphere separating two copies of the tangle of Figure 2.9(a) which however is not totally geodesic. Note that examples in the previous chapter show that this conjecture is false for two–component links. Indeed, as pointed out previously the Menasco-Reid conjecture is false for links with two or more components, by examples of Leininger [26].

If one considers immersed non–embedded totally geodesic surfaces, infinitely many (both compact and noncompact) may be found in the Figure 8 knot complement, by work of Maclachlan ([29], cf. [30], §9.6). His construction uses arithmeticity of the Figure 8 knot complement in a crucial way, and since this is the unique arithmetic knot complement ([46]) it is not necessarily a good indicator of the situation for hyperbolic knot complements in general. Indeed, work of Calegari shows that some hyperbolic knot complements in S^3, for example that of 8_{20}, contain no totally geodesic surfaces at all [11]. On the other hand, each twist knot complement contains an essential immersed 3–punctured sphere, which must be totally geodesic since the 3–punctured sphere is rigid. However, forthcoming work of Agol–Rafalski shows that there are at most finitely many other hyperbolic knot complements containing such a 3–punctured sphere.

4.2 Bounding

In attempting to construct manifolds containing totally geodesic surfaces, questions of bounding naturally arise, as the hyperbolic structures on the geodesic boundaries of the pieces obtained by cutting along a totally geodesic surface must be identical. Of particular interest are *highly symmetric* boundary structures, those which regularly cover a triangle orbifold, for these offer many isometries with which to construct twisted doubles. There are many questions about highly symmetric surfaces. Here is a classical one.

Question. *Does Klein's quartic curve bound?*

Klein's quartic curve is the *maximally* symmetric genus 3 surface, a regular cover of the $(2,3,7)$ triangle group, and this question asks if it occurs as the (sole) totally geodesic boundary component of a hyperbolic 3-manifold. Zimmermann has constructed a hyperbolic 3-manifold whose boundary consists of four copies of Klein's quartic curve [51]. The following conjecture addresses a situation very closely related to the subject of this thesis.

Conjecture. *The geodesic boundary of a hyperbolic tangle in the ball is never highly symmetric.*

A two–string tangle providing a counterexample to this conjecture would also furnish a counterexample to the conjecture of the previous section, by taking a double of the tangle twisted by an isometry with the correct action on endpoints of the tangle strings. We note that neither of the tangles of Section 2.4 of this thesis have highly symmetric boundary, although in both cases the boundary covers the modular surface. These are in fact the only tangle complements in the ball for which the author knows the totally geodesic boundary structure. On the other hand, the manifold N_a of Chapter 3 has highly symmetric boundary a 4–punctured sphere and two annular cusps, but the result of filling these annular cusps has nontrivial (although finite) fundamental group, and so N_a is not the complement of a tangle in the ball.

4.3 Volumes

A remark of Thurston from his notes [50] asserts, "One gets the feeling that volume is a very good measure of the complexity of a link *complement*, and that the ordinal structure is really inherent in 3-manifolds." There has subsequently been a wide–ranging project to classify small–volume 3-manifolds and orbifolds. The Figure 8 knot complement is now known to be the cusped orientable three manifold of smallest volume [12], and the 6 smallest volume cusped 3-orbifolds have been classified as well [36], [2]. The closed case has proven more resistant, although progress has been made (see eg. [44], [8], [9]) toward proving the conjecture that the "Weeks manifold" has smallest volume among closed orientable hyperbolic 3-manifolds. These results give evidence for the broad spirit of Thurston's assertion, that most reasonable measures of "complexity" of hyperbolic manifolds are minimized by minimal–volume examples.

Similar questions have been asked, and in some cases answered, about hyperbolic 3-manifolds with totally geodesic boundary. Kojima-Miyamoto showed that the minimal–volume compact hyperbolic 3-manifolds with geodesic boundary are obtained by identifying faces of two regular truncated tetrahedra, [25], and Miyamoto showed that the smallest hyperbolic manifolds with geodesic boundary are noncompact, obtained by gluing the faces of a regular ideal octahedron in pairs [37] (the manifold of Theorem 9 is one such example). Although Miyamoto's examples have annular cusps abutting the boundary, we do not know what is the smallest volume hyperbolic 3-manifold with geodesic boundary and a closed cusp. We suspect, however, that we have already found it.

Conjecture. *The smallest volume hyperbolic 3-manifold with totally geodesic boundary and a closed cusp is the complement of the tangle of Figure 2.9(b).*

This conjecture follows the spirit of Thurston's assertion, since the complement of a neighborhood of the tangle in Figure 2.9(b) is a genus 2 compression body. Furthermore the tangle itself has the lowest crossing number among all non-trivial tangles with a closed loop. We may also ask the corresponding question for manifolds with compact boundary.

Question. *What is the smallest volume hyperbolic 3-manifold with a closed cusp and compact geodesic boundary?*

The smallest such example of which we know was discovered by Frigerio-Martelli-Petronio, and is a knot complement in a genus 2 handlebody (see Figure 1 of [19]) with volume approximately 7.797637. The results of [19] give a "census" of all hyperbolic manifolds with compact geodesic boundary constructed from at most four partially truncated tetrahedra, showing in particular that this manifold is the only cusped example constructed from 3 or fewer (in fact it is constructed from 3). Following the spirit of Thurston's assertion again, it seems reasonable to conjecture that this is in fact the minimum volume example.

4.4 Arithmeticity

A hyperbolic three-orbifold of finite volume $M = \mathbb{H}^3/\Gamma$ is *arithmetic* if $k\Gamma$ has exactly one pair of complex–conjugate embeddings into $\mathbb{C}$, Γ has integral traces, and the *invariant quaternion algebra* is ramified at all real embeddings of $k\Gamma$. (We have not dealt with the invariant quaternion algebra in this thesis, but it is a commensurability invariant that may be associated to any nonelementary Kleinian group, see Chapter 3 of [30] for an overview.) The examples O_3, O_4, and O_2' of this thesis are all hyperbolic orbifolds with geodesic boundary satisfying the criteria above. We will call such orbifolds-with-boundary *subarithmetic*. The following question thus arises naturally in this context.

Question. *Does every finite–volume subarithmetic hyperbolic 3-orbifold with geodesic boundary embed isometrically in a finite–volume (boundaryless) arithmetic 3-orbifold?*

Given a subarithmetic 3-orbifold with geodesic boundary, its double would seem to be the most natural candidate to answer the question above in the positive. In particular, the double has the same invariant trace field and quaternion algebra as the original orbifold with geodesic boundary. However, as we remark below Lemma 6, the integral traces criterion fails for the double of O'_2. In fact it can be shown that the doubles of O'_3 and O'_4 have nonintegral traces as well. All three of these orbifolds have a *twisted* double which is arithmetic, though. On the other hand, as shown in Section 2.4, the doubles of O_∞ and O'_∞ are both arithmetic.

We remark that it follows from general considerations that the Kleinian group Γ associated to a subarithmetic hyperbolic 3-orbifold O with geodesic boundary is a subgroup of an arithmetic lattice $\Gamma' < \mathrm{PSL}_2(\mathbb{C})$; that is, a discrete group of finite covolume. This induces an immersion of O into $M = \mathbb{H}^3/\Gamma'$, and if $\Gamma < \Gamma'$ is *separable* — equal to the intersection of all finite–index subgroups of Γ' which contain it — then it follows from work of Scott [48] that O embeds in some finite cover of M. In general it is not known which subgroups of hyperbolic manifold groups are separable, although this topic is of considerable research interest (see eg. [27], [7]). Here it seems reasonable to conjecture that subgroups arising as above are separable, and thus that the above question has a positive answer.

Bibliography

[1] Colin Adams and Eric Schoenfeld. Totally geodesic Seifert surfaces in hyperbolic knot and link complements. I. *Geom. Dedicata*, 116:237–247, 2005.

[2] Colin C. Adams. Noncompact hyperbolic 3-orbifolds of small volume. In *Topology '90 (Columbus, OH, 1990)*, volume 1 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 1–15. de Gruyter, Berlin, 1992.

[3] Colin C. Adams. Toroidally alternating knots and links. *Topology*, 33(2):353–369, 1994.

[4] Colin C. Adams, Hanna Bennett, Christopher Davis, Michael Jennings, Jennifer Novak, Nicholas Perry, and Eric Schoenfeld. Totally geodesic Seifert surfaces in hyperbolic knot and link complements. II. arXiv:math.GT/0411358.

[5] Colin C. Adams, Jeffrey F. Brock, John Bugbee, and et al. Almost alternating links. *Topology Appl.*, 46(2):151–165, 1992.

[6] Colin C. Adams and Alan W. Reid. Quasi-Fuchsian surfaces in hyperbolic knot complements. *J. Austral. Math. Soc. Ser. A*, 55(1):116–131, 1993.

[7] I. Agol, D. D. Long, and A. W. Reid. The Bianchi groups are separable on geometrically finite subgroups. *Ann. of Math. (2)*, 153(3):599–621, 2001.

[8] Ian Agol. Volume change under drilling. *Geom. Topol.*, 6:905–916 (electronic), 2002.

[9] Ian Agol, Peter A. Storm, and William P. Thurston. Lower bounds on volumes of hyperbolic Haken 3-manifolds. arXiv:math.DG/0506338. with an appendix by Nathan Dunfield.

[10] Riccardo Benedetti and Carlo Petronio. *Lectures on hyperbolic geometry*. Universitext. Springer-Verlag, Berlin, 1992.

[11] Danny Calegari. Real places and torus bundles. *Geom. Dedicata*, 118:209–227, 2006.

[12] Chun Cao and G. Robert Meyerhoff. The orientable cusped hyperbolic 3-manifolds of minimum volume. *Invent. Math.*, 146(3):451–478, 2001.

[13] Eric Chesebro and Jason DeBlois. Trace fields and commensurability of link complements. In preparation.

[14] Daryl Cooper, Craig D. Hodgson, and Steven P. Kerckhoff. *Three-dimensional orbifolds and cone-manifolds*, volume 5 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2000. With a postface by Sadayoshi Kojima.

[15] David Coulson, Oliver A. Goodman, Craig D. Hodgson, and Walter D. Neumann. Computing arithmetic invariants of 3-manifolds. *Experiment. Math.*, 9(1):127–152, 2000.

[16] Jason DeBlois. Totally geodesic surfaces and homology. *Algebr. Geom. Topol.*, 6:1413–1428 (electronic), 2006.

[17] R. Frigerio. An infinite family of hyperbolic graph complements in S^3. *J. Knot Theory Ramifications*, 14(4):479–496, 2005.

[18] Roberto Frigerio. Hyperbolic manifolds with geodesic boundary which are determined by their fundamental group. *Topology Appl.*, 145(1-3):69–81, 2004.

[19] Roberto Frigerio, Bruno Martelli, and Carlo Petronio. Small hyperbolic 3-manifolds with geodesic boundary. *Experiment. Math.*, 13(2):171–184, 2004.

[20] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[21] Jonathan Hillman. *Algebraic invariants of links*, volume 32 of *Series on Knots and Everything.* World Scientific Publishing Co. Inc., River Edge, NJ, 2002.

[22] Kazuhiro Ichihara and Makoto Ozawa. Accidental surfaces in knot complements. *J. Knot Theory Ramifications*, 9(6):725–733, 2000.

[23] Akio Kawauchi. *A survey of knot theory.* Birkhäuser Verlag, Basel, 1996. Translated and revised from the 1990 Japanese original by the author.

[24] Sadayoshi Kojima. Deformations of hyperbolic 3-cone-manifolds. *J. Differential Geom.*, 49(3):469–516, 1998.

[25] Sadayoshi Kojima and Yosuke Miyamoto. The smallest hyperbolic 3-manifolds with totally geodesic boundary. *J. Differential Geom.*, 34(1):175–192, 1991.

[26] Christopher J. Leininger. Small curvature surfaces in hyperbolic 3-manifolds. *J. Knot Theory Ramifications*, 15(3):379–411, 2006.

[27] D. D. Long. Immersions and embeddings of totally geodesic surfaces. *Bull. London Math. Soc.*, 19(5):481–484, 1987.

[28] María Teresa Lozano and Józef H. Przytycki. Incompressible surfaces in the exterior of a closed 3-braid. I. Surfaces with horizontal boundary components. *Math. Proc. Cambridge Philos. Soc.*, 98(2):275–299, 1985.

[29] C. Maclachlan. Fuchsian subgroups of the groups $\mathrm{PSL}_2(O_d)$. In *Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984)*, volume

112 of *London Math. Soc. Lecture Note Ser.*, pages 305–311. Cambridge Univ. Press, Cambridge, 1986.

[30] Colin Maclachlan and Alan W. Reid. *The arithmetic of hyperbolic 3-manifolds*, volume 219 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003.

[31] Bernard Maskit. On Poincaré's theorem for fundamental polygons. *Advances in Math.*, 7:219–230, 1971.

[32] Bernard Maskit. *Kleinian groups*, volume 287 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1988.

[33] Hiroshi Matsuda. Complements of hyperbolic knots of braid index four contain no closed embedded totally geodesic surfaces. *Topology Appl.*, 119(1):1–15, 2002.

[34] John P. Mayberry and Kunio Murasugi. Torsion-groups of abelian coverings of links. *Trans. Amer. Math. Soc.*, 271(1):143–173, 1982.

[35] William Menasco and Alan W. Reid. Totally geodesic surfaces in hyperbolic link complements. In *Topology '90 (Columbus, OH, 1990)*, volume 1 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 215–226. de Gruyter, Berlin, 1992.

[36] Robert Meyerhoff. Sphere-packing and volume in hyperbolic 3-space. *Comment. Math. Helv.*, 61(2):271–278, 1986.

[37] Yosuke Miyamoto. Volumes of hyperbolic manifolds with geodesic boundary. *Topology*, 33(4):613–629, 1994.

[38] John W. Morgan. On Thurston's uniformization theorem for three-dimensional manifolds. In *The Smith conjecture (New York, 1979)*, volume 112 of *Pure Appl. Math.*, pages 37–125. Academic Press, Orlando, FL, 1984.

[39] Takeshi Morokuma. A characterization of fundamental domains of discontinuous groups acting on real hyperbolic spaces. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 25(2):157–183, 1978.

[40] Walter D. Neumann and Alan W. Reid. Amalgamation and the invariant trace field of a Kleinian group. *Math. Proc. Cambridge Philos. Soc.*, 109(3):509–515, 1991.

[41] Ulrich Oertel. Closed incompressible surfaces in complements of star links. *Pacific J. Math.*, 111(1):209–230, 1984.

[42] Luisa Paoluzzi and Bruno Zimmermann. On a class of hyperbolic 3-manifolds and groups with one defining relation. *Geom. Dedicata*, 60(2):113–123, 1996.

[43] Joan Porti. Mayberry-Murasugi's formula for links in homology 3-spheres. *Proc. Amer. Math. Soc.*, 132(11):3423–3431 (electronic), 2004.

[44] Andrew Przeworski. A universal upper bound on density of tube packings in hyperbolic space. *J. Differential Geom.*, 72(1):113–127, 2006.

[45] John G. Ratcliffe. *Foundations of hyperbolic manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

[46] Alan W. Reid. Arithmeticity of knot complements. *J. London Math. Soc. (2)*, 43(1):171–184, 1991.

[47] Makoto Sakuma. The homology groups of abelian coverings of links. *Math. Sem. Notes Kobe Univ.*, 7(3):515–530, 1979.

[48] Peter Scott. Subgroups of surface groups are almost geometric. *J. London Math. Soc. (2)*, 17(3):555–565, 1978.

[49] D. W. Sumners. On the homology of finite cyclic coverings of higher-dimensional links. *Proc. Amer. Math. Soc.*, 46:143–149, 1974.

[50] William P. Thurston. The geometry and topology of 3-manifolds. mimeographed lecture notes, 1979.

[51] Bruno Zimmermann. A note on surfaces bounding hyperbolic 3-manifolds. *Monatsh. Math.*, 142(3):267–273, 2004.

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